MAXIMAL SPACELIKE SURFACES IN A CERTAIN HOMOGENEOUS LORENTZIAN 3-MANIFOLD

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Abstract. The 2-parameter family of certain homogeneous Lorentzian 3-manifolds, which includes Minkowski 3-space and anti-de Sitter 3-space, is considered. Each homogeneous Lorentzian 3-manifold in the 2-parameter family has a solvable Lie group structure with left invariant metric. A generalized integral representation formula for maximal spacelike surfaces in the homogeneous Lorentzian 3-manifolds is obtained. The normal Gauß map of maximal spacelike surfaces and its harmonicity are discussed.

Introduction

In [7]-[8], J. Inoguchi studied Weierstraß-Enneper formula for minimal surfaces in the 2-parameter family of Riemannian homogeneous spaces $(\mathbb{R}^3, g[\mu_1, \mu_2])$ with

$$g[\mu_1, \mu_2] = e^{-\mu_1 t} dx^2 + e^{-\mu_2 t} dy^2 + dt^2.$$

Here, μ_1, μ_2 are real constants. Every homogeneous Riemannian manifold in this family can be represented as a solvable matrix Lie group with left invariant metric. This family of homogeneous Riemannian manifolds includes Euclidean 3-space and hyperbolic 3-space. Euclidean 3-space and hyperbolic 3-space are in fact the only homogeneous Riemannian manifolds in this family that have constant sectional curvature. The Weierstraß-Enneper formula obtained by Inoguchi is a generalized one that includes representation formulas for minimal surfaces in Euclidean 3-space, the well-known classical formula, and for minimal surfaces in hyperbolic 3-space obtained by M. Kokubu in [11] and independently by C. C. Góes and P. A. Q. Simões in [4]. The

2010 *Mathematics Subject Classification.* 53A10, 53C30, 53C42, 53C50.

Key words and phrases. Anti-de Sitter space, harmonic map, homogeneous manifold, Lorentzian manifold, maximal surface, Minkowski space, spacelike surface, solvable Lie group.

generalized Weierstraß-Enneper formula also contains an integral representation formula, obtained by Mercuri, Montaldo and Piu [14], for minimal surfaces in the Riemannian direct product $\mathbb{H}^2 \times \mathbb{E}^1$ of hyperbolic 2-space and the real line $\mathbb{E}^1$. Minimal surfaces in $\mathbb{H}^2 \times \mathbb{E}^1$ were also studied by B. Nelli and H. Rosenberg in [15] and [16]. On the other hand, in [12], the author considered the 2-parameter family of homogeneous Lorentzian 3-manifolds $(\mathbb{R}^3(x^0, x^1, x^2), g_{(\mu_1,\mu_2)})$ with Lorentzian metric

$$g_{(\mu_1,\mu_2)} = -(dx^0)^2 + e^{-2\mu_1 x^0}(dx^1)^2 + e^{-2\mu_2 x^0}(dx^2)^2.$$

Every homogeneous Lorentzian 3-manifold in this family can be represented as a solvable matrix Lie group with left invariant metric. This family of homogeneous Lorentzian 3-manifolds includes Minkowski 3-space $\mathbb{E}^3_1$, de Sitter 3-space $\mathbb{S}^3_1(c^2)$ of constant sectional curvature c^2, and $\mathbb{S}^2_1(c^2) \times \mathbb{E}^1$, the direct product of de Sitter 2-space $\mathbb{S}^2_1(c^2)$ of constant curvature c^2 and the real line $\mathbb{E}^1$. (In the family, only Minkowski 3-space and de Sitter 3-space have constant sectional curvature.) These three spaces may be considered as Lorentzian counterparts of Euclidean 3-space $\mathbb{E}^3$, hyperbolic 3-space $\mathbb{H}^3(-c^2)$, and the direct product $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$, respectively, of Thurston's eight model geometries [17]. In [12], the author obtained a generalized integral representation formula that includes Weierstraß representation formula for maximal spacelike surfaces in Minkowski 3-space studied independently by O. Kobayashi [10] and by L. McNertney [13], and Weierstraß representation formula for maximal spacelike surfaces in de Sitter 3-space.

In this paper, we consider the 2-parameter family of homogeneous Lorentzian 3-manifolds $(\mathbb{R}^3(x^0, x^1, x^2), g_{(\mu_1,\mu_2)})$ with Lorentzian metric

$$g_{(\mu_1,\mu_2)} = -e^{-2\mu_1 x^2}(dx^0)^2 + e^{-2\mu_2 x^2}(dx^1)^2 + (dx^2)^2.$$

Every homogeneous Lorentzian manifold in this family can also be represented as a solvable matrix Lie group with left invariant metric. This family of homogeneous Lorentzian 3-manifolds includes Minkowski 3-space $\mathbb{E}^3_1$, anti-de Sitter 3-space $\mathbb{H}^3_1(-c^2)$ of constant sectional curvature $-c^2$, $\mathbb{H}^2(-c^2) \times \mathbb{E}^1_1$, the direct product of hyperbolic plane $\mathbb{H}^2(-c^2)$ of constant curvature $-c^2$ and the timeline $\mathbb{E}^1_1$, and $\mathbb{H}^2_1(-c^2) \times \mathbb{E}^1$, the direct product of anti-de Sitter 2-space $\mathbb{H}^2_1(-c^2)$ of constant curvature $-c^2$ and the real line $\mathbb{E}^1$. (In the family, only Minkowski 3-space and anti-de Sitter 3-space have constant sectional curvature.) These four spaces may be considered as Lorentzian counterparts of Euclidean 3-space $\mathbb{E}^3$, 3-sphere $\mathbb{S}^3$, the direct product $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$, and $\mathbb{S}^2 \times \mathbb{E}^1$, the direct product of 2-sphere $\mathbb{S}^2$ and the real line $\mathbb{E}^1$, respectively, of Thurston's eight model geometries [17]. We obtain a generalized integral representation formula that includes, in particular, representation formulas for maximal spacelike surfaces in Minkowski 3-space ([10], [13]) and in anti-de Sitter 3-space. The normal Gauß map of maximal spacelike surfaces in $G(\mu_1, \mu_2)$ is discussed. It is shown that Minkowski 3-space $G(0, 0)$, anti-de Sitter 3-space $G(c, c)$, and $G(c, -c)$ are the only homogeneous Lorentzian 3-manifolds among the 2-parameter family members $G(\mu_1, \mu_2)$ in which the (projected) normal Gauß map of maximal spacelike surfaces is harmonic. The harmonic map equations for those cases are also obtained. In [2], J.H.S. de Lira and J. A. Hinojosa studied maximal surfaces in anti-de Sitter space viewed as the Lie group $\mathrm{SU}(1, 1)$ endowed with a bi-invariant metric. They showed that the projected Gauß map of a maximal surface in $\mathrm{SU}(1, 1)$ is harmonic and obtained a representation of maximal surfaces in $\mathrm{SU}(1, 1)$ in terms of the projected Gauß map.

1. Solvable Lie group

In this section, we study the following two-parameter family of homogeneous Lorentzian 3-manifolds;

$$\left\{(\mathbb{R}^3(x^0, x^1, x^2), g_{(\mu_1,\mu_2)}) \mid (\mu_1, \mu_2) \in \mathbb{R}^2\right\}, \tag{1}$$

where the metrics $g(\mu_1, \mu_2)$ are defined by

$$g(\mu_1, \mu_2) := -e^{-2\mu_1 x^2}(dx^0)^2 + e^{-2\mu_2 x^2}(dx^1)^2 + (dx^2)^2. \tag{2}$$

Proposition 1. *Each homogeneous space $(\mathbb{R}^3, g_{(\mu_1,\mu_2)})$ is isometric to the following solvable matrix Lie group:*

$$G(\mu_1, \mu_2) = \left\{ \begin{pmatrix} e^{\mu_1 x^2} & 0 & 0 & x^0 \\ 0 & e^{\mu_2 x^2} & 0 & x^1 \\ 0 & 0 & 1 & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x^0, x^1, x^2 \in \mathbb{R} \right\}$$

with left invariant metric. The group operation on $G(\mu_1, \mu_2)$ is the ordinary matrix multiplication and the corresponding group operation on $(\mathbb{R}^3, g_{(\mu_1,\mu_2)})$ is given by

$$(x^0, x^1, x^2) \cdot (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2) = (x^0 + e^{\mu_1 x^2}\tilde{x}^0, x^1 + e^{\mu_2 x^2}\tilde{x}^1, x^2 + \tilde{x}^2).$$

Proof. For $\tilde{a} = (a^0, a^1, a^2) \in G(\mu_1, \mu_2)$, denote by $L_{\tilde{a}}$ the left translation by $\tilde{a}$. Then

$$\begin{aligned} L_{\tilde{a}}(x^0, x^1, x^2) &= (a^0, a^1, a^2) \cdot (x^0, x^1, x^2) \\ &= (a^0 + e^{\mu_1 a^2} x^0, a^1 + e^{\mu_2 a^2} x^1, a^2 + x^2) \end{aligned}$$

and

$$\begin{aligned} L^*_{\tilde{a}} g_{(\mu_1,\mu_2)} = &-e^{-2\mu_1(a^2+x^2)}\{d(a^0 + e^{\mu_1 a^2} x^0)\}^2 + \\ &e^{-2\mu_2(a^2+x^2)}\{d(a^1 + e^{\mu_2 a^2} x^1)\}^2 + \{d(a^2 + x^2)\}^2 \\ &= -e^{-2\mu_1 x^2}(dx^0)^2 + e^{-2\mu_2 x^2}(dx^1)^2 + (dx^2)^2. \end{aligned}$$

This completes the proof. □

The Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$ is given explicitly by

$$\mathfrak{g}(\mu_1, \mu_2) = \left\{ \begin{pmatrix} \mu_1 y^2 & 0 & 0 & y^0 \\ 0 & \mu_2 y^2 & 0 & y^1 \\ 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,\middle|\, y^0, y^1, y^2 \in \mathbb{R} \right\}. \tag{3}$$

Then we can take the following orthonormal basis $\{E_0, E_1, E_2\}$ of $\mathfrak{g}(\mu_1, \mu_2)$:

$$E_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

Then the commutation relation of $\mathfrak{g}(\mu_1, \mu_2)$ is given by

$$\begin{aligned} &[E_0, E_1] = 0, \ [E_1, E_2] = -\mu_2 E_1, \\ &[E_2, E_0] = \mu_1 E_0. \end{aligned}$$

$[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$, so $\mathfrak{g}(\mu_1, \mu_2)$ is a solvable Lie algebra i.e. $G(\mu_1, \mu_2)$ is a solvable Lie group. For $X \in \mathfrak{g}(\mu_1, \mu_2)$, denote by $\mathrm{ad}(X)^*$ the *adjoint* operator of $\mathrm{ad}(X)$. Then it satisfies the equation

$$\langle [X, Y], Z \rangle = \langle Y, \mathrm{ad}(X)^*(Z) \rangle$$

for any $Y, Z \in \mathfrak{g}(\mu_1, \mu_2)$. Let U be the symmetric bilinear operator on $\mathfrak{g}(\mu_1, \mu_2)$ defined by

$$U(X, Y) := \frac{1}{2}\{\mathrm{ad}(X)^*(Y) + \mathrm{ad}(Y)^*(X)\}.$$

Lemma 2. *Let $\{E_0, E_1, E_2\}$ be the orthonormal basis for $\mathfrak{g}(\mu_1, \mu_2)$ defined in* (4)*. Then*

$$U(E_0, E_0) = \mu_1 E_2,\ U(E_1, E_1) = -\mu_2 E_2,\ U(E_2, E_2) = 0,$$
$$U(E_0, E_1) = 0,\ U(E_1, E_2) = \frac{\mu_2}{2} E_1,\ U(E_2, E_0) = \frac{\mu_1}{2} E_0.$$

Lemma 3 (M. Kokubu [11], K. Uhlenbeck [18])**.** *Let $\mathfrak{D}$ be a simply connected domain. A smooth map $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ is harmonic if and only if*

(5)
$$(\varphi^{-1}\varphi_u)_u + (\varphi^{-1}\varphi_v)_v - \mathrm{ad}(\varphi^{-1}\varphi_u)^*(\varphi^{-1}\varphi_u) - \mathrm{ad}(\varphi^{-1}\varphi_v)^*(\varphi^{-1}\varphi_v) = 0$$

holds.

Let $z = u + iv$. Then in terms of complex coordinates $z, \bar{z}$, the harmonic map equation (5) can be written as

$$\frac{\partial}{\partial \bar{z}}\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) - 2U\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}, \varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) = 0. \tag{6}$$

Let $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$. Then the equation (6) is equivalent to

$$A_{\bar{z}} + \bar{A}_z = 2U(A, \bar{A}). \tag{7}$$

The Maurer-Cartan equation is given by

$$A_{\bar{z}} - \bar{A}_z = [A, \bar{A}]. \tag{8}$$

The equations (7) and (8) can be combined to a single equation

$$A_{\bar{z}} = U(A, \bar{A}) + \frac{1}{2}[A, \bar{A}]. \tag{9}$$

The equation (9) is both the integrability condition for the differential equation $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$ and the condition for φ to be a harmonic map.

Left-translating the basis $\{E_0, E_1, E_2\}$, we obtain the following orthonormal frame field:

$$e_0 = e^{\mu_1 x^2}\frac{\partial}{\partial x^0},\ e_1 = e^{\mu_2 x^2}\frac{\partial}{\partial x^1},\ e_2 = \frac{\partial}{\partial x^2}.$$

The Levi-Civita connection ∇ of $G(\mu_1, \mu_2)$ is computed to be

$$\nabla_{e_0} e_0 = -\mu_1 e_2,\ \nabla_{e_0} e_1 = 0,\ \nabla_{e_0} e_2 = -\mu_1 e_0,$$
$$\nabla_{e_1} e_0 = 0,\ \nabla_{e_1} e_1 = \mu_2 e_2,\ \nabla_{e_1} e_2 = -\mu_2 e_1,$$
$$\nabla_{e_2} e_0 = 0,\ \nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = 0.$$

Let $K(e_i, e_j)$ denote the sectional curvature of $G(\mu_1, \mu_2)$ with respect to the tangent plane spanned by e_i and e_j for $i, j = 0, 1, 2$. Then

$$\begin{aligned} K(e_0, e_1) &= g^{00} R^1_{010} = -\mu_1\mu_2, \\ K(e_1, e_2) &= g^{11} R^2_{121} = -\mu_2^2, \\ K(e_0, e_2) &= g^{00} R^2_{020} = -\mu_1^2, \end{aligned} \tag{10}$$

where $g_{ij} = g_{(\mu_1,\mu_2)}(e_i, e_j)$ denotes the metric tensor of $G(\mu_1, \mu_2)$. Hence, we see that $G(\mu_1, \mu_2)$ has a constant sectional curvature if and only if $\mu_1^2 = \mu_2^2 = \mu_1\mu_2$. If $c := \mu_1 = \mu_2$, then $G(\mu_1, \mu_2)$ is locally isometric to $\mathbb{H}^3_1(-c^2)$, the anti-de Sitter 3-space of constant sectional curvature $-c^2$. (See Example 2.) If $G(\mu_1, \mu_2)$ has a constant sectional curvature and $\mu_1 = -\mu_2$, then $\mu_1 = \mu_2 = 0$, so $G(\mu_1, \mu_2) = G(0, 0) \cong \mathbb{E}^3_1$ (Example 1).

Example 1. (Minkowski 3-space) The Lie group $G(0, 0)$ is isomorphic and isometric to the Minkowski 3-space $\mathbb{E}^3_1$ i.e., $\mathbb{R}^3(x^0, x^1, x^2)$ with the metric $-(dx^0)^2 + (dx^1)^2 + (dx^2)^2$.

Example 2. (Anti-de Sitter 3-space) Take $\mu_1 = \mu_2 = c \neq 0$. Then $G(c, c)$ is

$$(\mathbb{R}^3(x^0, x^1, x^2), e^{-2cx^2}\{-(dx^0)^2 + (dx^1)^2\} + (dx^2)^2).$$

$G(c, c)$ is closely related to the anti-de Sitter 3-space. To see this in details, let $\mathbb{E}^4_2$ be the pseudo-Euclidean 4-space with the metric $-(du^0)^2 - (du^1)^2 + (du^2)^2 + (du^3)^2$ in terms of rectangular coordinate system (u^0, u^1, u^2, u^3). The *anti-de Sitter* 3-*space* $\mathbb{H}^3_1(-c^2)$ of constant sectional curvature $-c^2$ is the hyperquadric in $\mathbb{E}^4_2$:

$$\mathbb{H}^3_1(-c^2) = \left\{(u^0, u^1, u^2, u^3) \in \mathbb{E}^4_2 : \ -(u^0)^2 - (u^1)^2 + (u^2)^2 + (u^3)^2 = -\frac{1}{c^2}\right\}.$$

This hyperquadric is isometrically diffeomorphic to the Lie group $\mathrm{SU}(1, 1)$ with a bi-invariant metric, which is the model of the anti-de Sitter 3-space J.H.S. de Lira and J. A. Hinojosa used for their study of maximal surfaces and their Gauß map [2]. Let

$$U = \{(u^0, u^1, u^2, u^3) \in \mathbb{H}^3_1(-c^2) : u^1 + u^2 > 0\}$$

and introduce a local coordinate system (x^0, x^1, x^2) on U by

$$\begin{aligned} x^0 &= \frac{u^0}{c(u^1+u^2)}, \\ x^1 &= \frac{u^3}{c(u^1+u^2)}, \\ x^2 &= -\frac{1}{c}\ln[c(u^1+u^2)]. \end{aligned}$$

The local coordinate chart U is identified with $G(c,c)$. Although it covers only half of the anti-de Sitter 3-space, since we are concerned with only local aspects of maximal surfaces in this paper, $G(c,c)$ will be considered as the anti-de Sitter space for the rest of this paper. $G(c,c)$ is called the *flat chart model* of the anti-de Sitter 3-space. While the author could not find any reference on the flat chart model of the anti-de Sitter space, the flat chart model of de Sitter space-time (de Sitter 4-space) can be found on page 125 of the book [5] by S. W. Hawking and G. F. R. Ellis, and it is the steady state model of the expanding universe proposed by Bondi and Gold [1] and Hoyle [6]. $G(c,c)$ can be seen as a warped product $\mathbb{E}^1 \times_f \mathbb{E}^2_1$ with warping function $f(x^2) = e^{-cx^2}$. Introducing $y^0 = cx^0$, $y^1 = cx^1$, and $y^2 = e^{cx^2}$, we also obtain the half-space model of the anti-de Sitter 3-space with an analogue of Poincaré metric

$$g_c := \frac{-(dy^0)^2 + (dy^1)^2 + (dy^2)^2}{c^2(y^2)^2}.$$

Example 3 (Direct Product $\mathbb{H}^2(-c^2) \times \mathbb{E}^1_1$). Take $(\mu_1, \mu_2) = (0, c)$ with $c \neq 0$. Then the resulting homogeneous spacetime is $\mathbb{R}^3$ with the Lorentzian metric

$$-(dx^0)^2 + e^{-2cx^2}(dx^1)^2 + (dx^2)^2.$$

$G(0,c)$ is identified with $\mathbb{H}^2(-c^2) \times \mathbb{E}^1_1$, the direct product of hyperbolic plane $\mathbb{H}^2(-c^2)$ of constant curvature $-c^2$ and the timeline $\mathbb{E}^1_1$.

Example 4 (Direct Product $\mathbb{H}^2_1(-c^2) \times \mathbb{E}^1$). Take $(\mu_1, \mu_2) = (c, 0)$ with $c \neq 0$. Then the resulting homogeneous spacetime is $\mathbb{R}^3$ with the Lorentzian metric

$$-e^{-2cx^2}(dx^0)^2 + (dx^2)^2 + (dx^1)^2.$$

$G(c,0)$ is identified with $\mathbb{H}^2_1(-c^2) \times \mathbb{E}^1$, the direct product of anti-de Sitter 2-space $\mathbb{H}^2_1(-c^2)$ of constant curvature $-c^2$ and the real line $\mathbb{E}^1$.

Example 5 (Homogeneous Spacetime $G(c,-c)$). Let $\mu_1 = c$ and $\mu_2 = -c$ with $c \neq 0$. Then the resulting homogeneous spacetime $G(c,-c)$ is $\mathbb{R}^3$ with the Lorentzian metric

$$-e^{-2cx^2}(dx^0)^2 + e^{2cx^2}(dx^1)^2 + (dx^2)^2.$$

2. Integral representation formula

In this section, we obtain a general integral representation formula for maximal spacelike surfaces in $G(\mu_1,\mu_2)$ analogously to [7] and [12].

Let $\mathfrak{D}(z,\bar{z})$ be a simply connected domain and $\varphi : \mathfrak{D} \longrightarrow G(\mu_1,\mu_2)$ a smooth map. If we write $\varphi(z) = (x^0(z), x^1(z), x^2(z))$ then by direct calculation

$$A = x^0_z e^{-\mu_1 x^2} E_0 + x^1_z e^{-\mu_2 x^2} E_1 + x^2_z E_2.$$

It follows from the harmonic map equation (7) that

Lemma 4. *φ is harmonic if and only if the following equations hold:*

$$\begin{aligned} x^0_{z\bar{z}} - \mu_1(x^0_{\bar{z}} x^2_z + x^0_z x^2_{\bar{z}}) &= 0, \\ x^1_{z\bar{z}} - \mu_2(x^1_{\bar{z}} x^2_z + x^1_z x^2_{\bar{z}}) &= 0, \\ x^2_{z\bar{z}} - \mu_1 x^0_z x^0_{\bar{z}} e^{-2\mu_1 x^2} + \mu_2 x^1_z x^1_{\bar{z}} e^{-2\mu_2 x^2} &= 0. \end{aligned}$$

The exterior derivative d is decomposed as

$$d = \partial + \bar{\partial},\ \partial = \frac{\partial}{\partial z} dz,\ \bar{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z},$$

with respect to the conformal structure of $\mathfrak{D}$. Let $\omega^0 = e^{-\mu_1 x^2} x^0_z dz$, $\omega^1 = e^{-\mu_2 x^2} x^1_z dz$, $\omega^2 = x^2_z dz$. Then by Lemma 4, the triplet $\{\omega^0, \omega^1, \omega^2\}$ of (1,0)-forms satisfies the following differential system:

$$\bar{\partial}\omega^i = \mu_{i+1} \overline{\omega^i} \wedge \omega^2,\ i = 0, 1, \tag{11}$$

$$\bar{\partial}\omega^2 = \mu_1 \overline{\omega^0} \wedge \omega^0 - \mu_2 \overline{\omega^1} \wedge \omega^1. \tag{12}$$

Proposition 5. *Let $\{\omega^0, \omega^1, \omega^2\}$ be a solution to (11)-(12) on a simply connected domain $\mathfrak{D}$. Then*

$$\varphi(z,\bar{z}) = 2\mathrm{Re} \int_{z_0}^{z} \left(e^{\mu_1 x^2(z,\bar{z})} \cdot \omega^0, e^{\mu_2 x^2(z,\bar{z})} \cdot \omega^1, \omega^2 \right)$$

is a harmonic map into $G(\mu_1,\mu_2)$.
Conversely, any harmonic map of $\mathfrak{D}$ into $G(\mu_1,\mu_2)$ can be represented in this form.

Corollary 6. *Let* $\{\omega^0, \omega^1, \omega^2\}$ *be a solution to* (11)-(12) *along with*

$$-\omega^0 \otimes \omega^0 + \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 = 0 \tag{13}$$

on a simply connected domain $\mathfrak{D}$. *Then*

$$\varphi(z, \bar{z}) = 2\mathrm{Re} \int_{z_0}^{z} \left(e^{\mu_1 x^2(z,\bar{z})} \cdot \omega^0, e^{\mu_2 x^2(z,\bar{z})} \cdot \omega^1, \omega^2 \right)$$

is a weakly conformal harmonic map into $G(\mu_1, \mu_2)$. *Moreover* $\varphi(z, \bar{z})$ *is a maximal spacelike surface*[1] *if*

$$-\omega^0 \otimes \overline{\omega^0} + \omega^1 \otimes \overline{\omega^1} + \omega^2 \otimes \overline{\omega^2} \neq 0.$$

3. The normal Gauss map

Let $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ be a conformal surface. Take the future-pointing unit normal N along φ. Then, by the left translation we obtain the following smooth map:

$$\varphi^{-1} \cdot N : \mathfrak{D} \longrightarrow \mathbb{H}^2(-1),$$

where

$$\mathbb{H}^2(-1) = \{u^0 E_0 + u^1 E_1 + u^2 E_2 : -(u^0)^2 + (u^1)^2 + (u^2)^2 = -1,\ u^0 > 0\} \subset \mathfrak{g}(\mu_1, \mu_2)$$

is the unit hyperbolic 2-space. The Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$ is identified with Minkowski 3-space $\mathbb{E}^3_1(u^0, u^1, u^2)$ via the orthonormal basis $\{E_0, E_1, E_2\}$. The smooth map $\varphi^{-1} \cdot N$ is called the *normal Gauß map* of φ.

Let $\varphi : \mathfrak{D} \to G(\mu_1, \mu_2)$ be a maximal spacelike immersion of a simply connected Riemann surface $\mathfrak{D}$ determined by the data $(\omega^0, \omega^1, \omega^2)$. Write the data as $\omega^i = \psi^i dz,\ i = 0, 1, 2$. Then the induced metric I of φ is

$$\begin{aligned} I &= 2(-\omega^0 \otimes \overline{\omega^0} + \omega^1 \otimes \overline{\omega^1} + \omega^2 \otimes \overline{\omega^2}) \\ &= 2(-|\psi^0|^2 + |\psi^1|^2 + |\psi^2|^2) dz d\bar{z}. \end{aligned} \tag{14}$$

From the conformality condition (13),

$$-(\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 = 0. \tag{15}$$

Hence, we can introduce two complex valued functions f and g by

$$f := \psi^1 - i\psi^2,\ g := \frac{\psi^0}{\psi^1 - i\psi^2}. \tag{16}$$

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[1] From here on we mean a surface by an immersion.

Using these two functions, φ can be written as

$$\varphi(z,\bar{z}) = 2\mathrm{Re}\int_{z_0}^{z}\left(e^{\mu_1 x^2} fg, \frac{1}{2}e^{\mu_2 x^2} f(1+g^2), \frac{i}{2}f(1-g^2)\right)dz. \tag{17}$$

$\varphi^{-1}\varphi_z$ is given by

$$\varphi^{-1}\varphi_z = fgE_0 + \frac{1}{2}f(1+g^2)E_1 + \frac{i}{2}f(1-g^2)E_2. \tag{18}$$

So, the first fundamental form[2] I is given in terms of f and g by

$$\begin{aligned} I &= 2\langle\varphi^{-1}\varphi_z, \varphi^{-1}\varphi_{\bar{z}}\rangle dz d\bar{z} \\ &= |f|^2(1-|g|^2)^2 dz d\bar{z}. \end{aligned} \tag{19}$$

The normal Gauß map is computed to be

$$\varphi^{-1}\cdot N = \frac{1}{1-|g|^2}[(1+|g|^2)E_0 + 2\mathrm{Re}\,(g)E_1 + 2\mathrm{Im}\,(g)E_2].$$

Let $\mathbb{D} = \{\zeta^1 E_1 + \zeta^2 E_2 \in \mathbb{R}^2 : (\zeta^1)^2 + (\zeta^2)^2 < 1\}$. Under the stereographic projection from $-E_0$

$$\wp^+ : \mathbb{H}^2(-1) \longrightarrow \mathbb{D};\ \wp^+(u^0E_0 + u^1E_1 + u^2E_2) = \frac{u^1}{1+u^0}E_1 + \frac{u^2}{1+u^0}E_2,$$

the map $\varphi^{-1}\cdot N$ is identified with the function g. If $\mathbb{H}^2(-1)$ is defined to be the hyperboloid of two sheets

$$\mathbb{H}^2(-1) = \{u^0E_0 + u^1E_1 + u^2E_2 : -(u^0)^2 + (u^1)^2 + (u^2)^2 = -1\},$$

then $\wp^+ : \mathbb{H}^2(-1) \longrightarrow \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the extended complex plane $\mathbb{C}\cup\{\infty\}$. The function g is called the *projected normal Gauß map* of φ. It follows from (11) and (12) that

$$\psi^i_{\bar{z}} = \mu_{i+1}\overline{\psi^i}\psi^2,\ i = 0,1, \tag{20}$$

$$\psi^2_{\bar{z}} = \mu_1|\psi^0|^2 - \mu_2|\psi^1|^2. \tag{21}$$

Using (20) and (21), we obtain

$$\frac{\partial f}{\partial \bar{z}} = -i|f|^2\left\{\mu_1|g|^2 - \frac{\mu_2}{2}(1+\bar{g}^2)\right\}, \tag{22}$$

$$\frac{\partial g}{\partial \bar{z}} = \frac{i}{2}\bar{f}\{\mu_1\bar{g}(1+g^2) - \mu_2 g(1+\bar{g}^2)\}. \tag{23}$$

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[2]It can be also obtained directly from (14).

As is seen in Section 1, $G(0,0)$ and $G(c,c)$ are the only cases of solvable Lie group $G(\mu_1,\mu_2)$ with constant sectional curvature. For $G(0,0)$,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} = 0,$$

that is, both f and g are holomorphic. From (18), we retrieve the Weierstraß representation formula for maximal spacelike surface $\varphi : \mathfrak{D} \longrightarrow G(0,0)$ in terms of holomorphic data (g,f), which was obtained independently by O. Kobayashi [10] and by L. McNertney [13]. For $G(c,c)$,

$$\frac{\partial f}{\partial \bar{z}} = -ic|f|^2 \left\{ |g|^2 - \frac{1}{2}(1+\bar{g}^2) \right\}, \tag{24}$$

$$\frac{\partial g}{\partial \bar{z}} = \frac{ic}{2}\bar{f}(\bar{g}-g)(1-|g|^2). \tag{25}$$

Then the Weierstraß representation formula (18) with $\mu_1 = \mu_2 = c$ gives rise to maximal spacelike surfaces in $G(c,c)$. From (25), the representation formula (18) can be written in terms of only the Gauß map g as

$$\varphi^{-1}\varphi_z = \frac{i\bar{g}_z}{c(\bar{g}-g)(1-|g|^2)}[2gE_0 + (1+g^2)E_1 + i(1-g^2)E_2] \tag{26}$$

which is analogous to the formula (81) of Theorem 4.2 in [2]. If g is holomorphic, it follows from (25) that $g = \bar{g}$ or $|g|^2 = 1$. If $|g|^2 = 1$ then we see from (19) that $I = 0$. If $g = \bar{g}$ then g is real. This means that $\psi^2 = 0$ (see (16)) and from the conformality condition (15) we get $(\psi^0)^2 = (\psi^1)^2$. But along with $\psi^2 = 0$ this also leads to $I = 0$. Hence the projected normal Gauß map of maximal spacelike surfaces in $\mathbb{H}^3_1(-c^2)$ cannot be holomorphic.

It follows from (22) and (23) that the projected normal Gauß map g satisfies the partial differential equation:

$$\begin{aligned} g_{z\bar{z}} &- \frac{(\mu_1^2-\mu_2^2)g(1+g^2)(1-\bar{g}^2)|g_{\bar{z}}|^2}{[\mu_1 g(1+\bar{g}^2)-\mu_2\bar{g}(1+g^2)][\mu_1\bar{g}(1+g^2)-\mu_2 g(1+\bar{g}^2)]} \\ &- \frac{2\mu_1|g|^2-\mu_2(1+\bar{g}^2)}{\mu_1\bar{g}(1+g^2)-\mu_2 g(1+\bar{g}^2)} g_z g_{\bar{z}} = 0. \end{aligned} \tag{27}$$

The equation (27) is not the harmonic map equation for the projected normal Gauß map g in general. The following theorem tells under what conditions it becomes the harmonic map equation for g.

Theorem 7. *The partial differential equation* (27) *is the harmonic map equation for g if and only if $\mu_1^2 = \mu_2^2$. If $\mu_1 = \mu_2 \neq 0$, then* (27) *is simplified to*

$$g_{z\bar{z}} + \frac{1+\bar{g}^2 - 2|g|^2}{(\bar{g}-g)(1-|g|^2)} g_z g_{\bar{z}} = 0. \tag{28}$$

This equation is the harmonic map equation for a map $g : \mathfrak{D}(z,\bar{z}) \longrightarrow \left(\hat{\mathbb{C}}(w,\bar{w}), \frac{2dwd\bar{w}}{|(\bar{w}-w)(1-|w|^2)|}\right)$. If $\mu_1 = -\mu_2$, then (27) *is simplified to*

$$g_{z\bar{z}} - \frac{1+\bar{g}^2 + 2|g|^2}{(g+\bar{g})(1+|g|^2)} g_z g_{\bar{z}} = 0. \tag{29}$$

This equation is the harmonic map equation for a map $g : \mathfrak{D}(z,\bar{z}) \longrightarrow \left(\hat{\mathbb{C}}(w,\bar{w}), \frac{2dwd\bar{w}}{|(w+\bar{w})(1+|w|^2)|}\right)$.

Proof. The *tension field* $\tau(g)$ of g is given by ([3], [19])

$$\tau(g) = 4\lambda^{-2}(g_{z\bar{z}} + \Gamma^w_{ww} g_z g_{\bar{z}}), \tag{30}$$

where λ is a parameter of conformality. Here, Γ^w_{ww} denotes the Christoffel symbols of $\hat{\mathbb{C}}(w,\bar{w})$. Comparing the equations (27) and $\tau(g) = 0$, we see that (27) is a harmonic map equation if and only if $\mu_1^2 = \mu_2^2$. In order to find a suitable metric on $\hat{\mathbb{C}}(w,\bar{w})$ with which (27) is a harmonic map equation, one simply needs to solve the first order partial differential equations

$$\Gamma^w_{ww} = \begin{cases} \dfrac{1+\bar{w}^2 - 2|w|^2}{(\bar{w}-w)(1-|w|^2)} & \text{if } \mu_1 = \mu_2 \neq 0, \\ \\ -\dfrac{1+\bar{w}^2 + 2|w|^2}{(w+\bar{w})(1+|w|^2)} & \text{if } \mu_1 = -\mu_2. \end{cases}$$

The solutions are

$$(g_{w\bar{w}}) = \begin{cases} \begin{pmatrix} 0 & \frac{1}{(\bar{w}-w)(1-|w|^2)} \\ \frac{1}{(\bar{w}-w)(1-|w|^2)} & 0 \end{pmatrix} & \text{if } \mu_1 = \mu_2 \neq 0, \\ \\ \begin{pmatrix} 0 & \frac{1}{(w+\bar{w})(1+|w|^2)} \\ \frac{1}{(w+\bar{w})(1+|w|^2)} & 0 \end{pmatrix} & \text{if } \mu_1 = -\mu_2, \end{cases}$$

respectively. □

Remark 1. It is well-known that the projected Gauß map g of a maximal spacelike surface in Minkowski 3-space $G(0,0)$ satisfies the Laplace-Beltrami equation

$$\triangle g = 4\lambda^{-2} g_{z\bar{z}} = 0.$$

Remark 2. Theorem 7 tells us that Minkowski 3-space $G(0,0)$, anti-de Sitter 3-space $G(c,c)$, and $G(c,-c)$ are the only homogeneous 3-spacetimes among $G(\mu_1,\mu_2)$ in which the projected normal Gauß map of a maximal spacelike surface is harmonic.

Remark 3. If g is not anti-holomorphic, using (24) and (25), the harmonic map equation (28) can be written as

$$g_{z\bar{z}} + \frac{f_{\bar{z}}}{f} g_z = 0. \tag{31}$$

The second-order linear differential equation (31) can be reduced to a first-order linear differential equation using the substitution $h = g_z$

$$h_{\bar{z}} + \frac{f_{\bar{z}}}{f} h = 0. \tag{32}$$

Its solution is $h(z,\bar{z}) = g_z(z,\bar{z}) = \frac{\eta(z)}{f(z,\bar{z})}$, where $\eta(z)$ is a holomorphic function.

Finding interesting examples of maximal spacelike surfaces in $G(c,c)$ and $G(c,-c)$ is a daunting task comparing with finding those in $G(0,0)$ where g and f are a meromorphic and a holomorphic functions, respectively. We leave it for a future discussion and conclude this paper with a simple example below.

Example 6. Let $\mathfrak{D} = \{z \in \mathbb{C} : \mathrm{Re} z < 0\}$. The Gauß map $g : \mathfrak{D} \longrightarrow \mathbb{D}$ given by $g_c(z,\bar{z}) = ae^{c\bar{z}}$ with $|a| < 1$ is an anti-holomorphic map, so it automatically satisfies (28). The maximal spacelike surface $\varphi_c : \mathfrak{D} \longrightarrow G(c,c)$ whose projected Gauß map is g_c satisfies

$$\varphi_c^{-1}(\varphi_c)_z = \frac{i\bar{a}}{(\bar{a} - ae^{-c(z-\bar{z})})(1-|a|^2 e^{c(z+\bar{z})})}[2ae^{c\bar{z}}E_0 + (1+a^2e^{2c\bar{z}})E_1 + i(1-a^2e^{2c\bar{z}})E_2].$$

As $c \to 0$, $g_c \to g_0 = a$, so the limit surface φ_0 is a spacelike plane. Indeed, as $c \to 0$, $\varphi_c^{-1}(\varphi_c)_z$ approaches

$$\varphi_0^{-1}(\varphi_0)_z = \frac{1}{(\bar{a}-a)(1-|a|^2)}[2i|a|^2 E_0 + i(a|a|^2 + \bar{a})E_1 + (a|a|^2 - \bar{a})E_2].$$

Conflict of Interest Statement: On behalf of all authors, the corresponding author states that there is no conflict of interest.

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