Proposition 1. *For any $n \times n$ real or complex matrix X,*

$$e^X := \sum_{m=0}^{\infty} \frac{X^m}{m!} \tag{1}$$

converges and is a continuous function.

Before we prove the proposition, let us recall the norm of a vector $x = (x_1, \cdots, x_n) \in \mathbb{C}^n$ is defined to be

$$||x|| = \sqrt{\langle x, x \rangle} = \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2}. \tag{2}$$

Define the norm of a matrix in the space $M_n(\mathbb{C})$ as $\mathbb{C}^{n^2}$ i.e. we define

$$||X|| \left(\sum_{k,l=1}^{n} |X_{kl}|^2 \right)^{1/2}. \tag{3}$$

This norm satisfies the inequaities

$$||X + Y|| \leq ||X|| + ||Y|| \tag{4}$$

$$||XY|| \leq ||X||||Y|| \tag{5}$$

for all $X, Y \in M_n(\mathbb{C})$. The inequality (4) is simply a triangle inequality and it can be shown by the triangle inequality for the norm (2). The inequality (5) may be viewed as a generalized Cauchy-Schwarz inequality and it can be shown by the Cauchy-Schwarz inequality for the norm (2). The norm (3) on $M_n(\mathbb{C})$ is called the *Hilbert-Schmidt norm*.

Now we are ready proove the proposition.

Proof. By the inequality (5), we have $||X^m|| \leq ||X||^m$, hence we obtain

$$\sum_{m=0}^{\infty} \left|\left| \frac{X^m}{m!} \right|\right| \leq \sum_{m=0}^{\infty} \frac{||X||^m}{m!} = e^{||X||} < \infty.$$

Therefore $e^X = \displaystyle\sum_{m=0}^{\infty} \frac{X^m}{m!}$ converges absolutely and so it converges.

For continuity, X^m is a continuous function of X so the partial sums

$$s_n := \sum_{m=0}^{n} \frac{X^m}{m!}$$

are also continuous. Note that the sequence of functions $\{s_n\}$ converges uniformly on the compact set $\{||X|| \leq R\}$ and hence the sum is continuous. $\square$