Number Theory
Problem Set 3
Congruences

1. Compute the addition and multiplication tables for the commutative ring with unity $\mathbb{Z}/6\mathbb{Z}$.

2. Is there an integer x such that $7x \equiv 1 \mod 17$? If so, name such an integer. What about $6x \equiv 1 \mod 15$?

   Answer. There is an integer x such that $7x \equiv 1 \mod 17$. For example $x = 5$ is such an integer. On the other hand, there is no integer x such that $6x \equiv 1 \mod 15$.

3. Describe all solutions of the following congruences:

   (a) $3x \equiv 4 \mod 7$
   Answer. $x \equiv 6 \mod 7$.

   (b) $3x \equiv 4 \mod 12$
   Answer. There is no solution.

   (c) $9x \equiv 12 \mod 21$
   Answer. The same as part (a).

   (d) $27x \equiv 25 \mod 256$
   Answer. $x \equiv 219 \mod 256$.

   (e) $27x \equiv 72 \mod 900$
   Answer. $x \equiv 36 \mod 100$.

   (f) $103x \equiv 612 \mod 676$
   Answer. $x \equiv 636 \mod 676$.

4. Prove that $n^5 - n$ is always divisible by 30.

   Hint. Prove separately that $n^5 - n$ is divisible by 2, 3, and 5.

5. Prove or disprove: If $a \equiv b \mod d$ and $d|n$, then $a \equiv b \mod n$.

6. Prove that if n is odd, then $2^{n-1} - 1 \equiv 0 \mod 3$.

   Hint: Since n is odd, $n = 2k + 1$ for $k \in \mathbb{Z}$. Use Fermat's Little Theorem where applicable.

7. Prove that if p is a prime, then $x \equiv \pm 1 \mod p$ are the only solutions of the equation $x^2 \equiv 1 \mod p$.

   Hint. Recall that $\mathbb{Z}/p\mathbb{Z}$ has no zero divisors.

8. Prove Wilson's Theorem: for any prime p, $(p-1)! \equiv -1 \mod p$. Prove that $(n-1)! \not\equiv -1 \mod n$ if n is not prime.

   Hint. For Wilson's Theorem, the problem 7 helps. For the second part, try to prove it by contradiction.

9. Find the smallest nonnegative soluion of each of the following systems of congruences.

   (a)

$$\begin{aligned} x &\equiv 2 \mod 3 \\ x &\equiv 3 \mod 5 \\ x &\equiv 4 \mod 11 \\ x &\equiv 5 \mod 16 \end{aligned}$$

   Answer. 1973.

   (b)

$$\begin{aligned} x &\equiv 12 \mod 31 \\ x &\equiv 87 \mod 127 \\ x &\equiv 91 \mod 255 \end{aligned}$$

   Answer. 63841.

   (c)

$$\begin{aligned} 19x &\equiv 103 \mod 900 \\ 10x &\equiv 511 \mod 841 \end{aligned}$$

   Answer. 58837.

10. Find $\phi(n)$ for all n from 90 to 100.

Answer.

n	90	91	92	93	94	95	96	97	98	99	100
$\phi(n)$	24	72	44	60	46	72	32	96	42	60	40