GROUP THEORY
PROBLEM SET 9
THE ISOMORPHISM THEOREMS, DIRECT PRODUCTS & FINITELY GENERATED ABELIAN GROUPS

(1) Let G be the group of all real-valued functions on $[0,1]$, where we define, for $f, g \in G$, addition by $(f+g)(x) = f(x)+g(x)$ for every $x \in [0,1]$. If $N = \{f \in G : f\left(\frac{1}{4}\right) = 0\}$, prove that $G/N \cong (\mathbb{R}, +)$.

(2) Let G be the group of nonzero real numbers under multiplication and let $N = \{1, -1\}$. Prove that $G/N \cong (\mathbb{R}^+, \cdot)$ where $\mathbb{R}^+$ is the set of all positive real numbers.

(3) If G_1, G_2 are two groups and $G = G_1 \times G_2 = \{(a,b) : a \in G_1,\ b \in G_2\}$, where we define $(a,b)(c,d) = (ac, bd)$, show that:
   (a) $N = \{(a, e_2) : a \in G_1\}$, where e_2 is the unit element of G_2, is a normal subgroup of G.
   (b) $N \cong G_1$.
   (c) $G/N \cong G_2$.

(4) If G is an abelian group of order $p_1 p_2 \cdots p_k$, where $p_1, p_2, \cdots, p_k$ are distinct primes, prove that G is cyclic.

(5) Find all abelian groups, up to isomorphism, of order $2^3 \cdot 3^4 \cdot 5^2$.

(6) Let G be a group and let $A, B \lhd G$ such that $G = AB$ and $A \cap B = \{e\}$. Then G is said to be the *internal direct product* of A and B.
   (a) Each element $g \in G$ is uniquely represented as $g = ab$ for some $a \in A$ and $b \in B$.
   (b) For any $a \in A$ and $b \in B$, $ab = ba$.
   (c) Show that $G \cong A \times B$.

(7) If G is an abelian group and if G has an element of order m and one of order n, where m and n are relatively prime, prove that G has an element of order mn.