GROUP THEORY
PROBLEM SET 7
NORMAL SUBGROUPS

(1) Recall that $N \leq G$ is a normal subgroup of G if $\forall g \in G$, $g^{-1}Ng \subset N$. Use this definition to prove that $N \leq G$ is a normal subgroup of G if and only if $\forall g \in G$, $gN = Ng$.
(2) Show that $Z(D_4) = \{1, \sigma^2\}$.
(3) Show that $H = \{1, \tau\}$ is not a normal subgroup of D_4.
(4) Prove that the center $Z(G)$ of a group G is a normal subgroup of G.
(5) Prove that if $N \leq Z(G)$ then $N \lhd G$.
(6) If $N \lhd G$ and $\varphi : G \longrightarrow G'$ is a homomorphism of G onto G', prove that $\varphi(N) \lhd G'$.
(7) If $N \lhd G$ and $M \lhd G$, then

$$MN = \{mn : m \in M, n \in N\}$$

is a normal subgroup of G.
(8) If $M \lhd G$ and $N \lhd G$, prove that $M \cap N \lhd G$.
(9) If H is a subgroup of G and $N = \bigcap_{a \in G} a^{-1}Ha$, prove that $N \lhd G$.
(10) If H is a subgroup of G, let $N(H) = \{a \in G : a^{-1}Ha = H\}$. Prove that:
  (a) $N(H)$ is a subgroup of G and $H \subset N(H)$.
  (b) $H \lhd N(H)$.
  (c) If K is a subgroup of G such that $H \lhd K$, then $K \subset N(H)$. This means that $N(H)$ is the largest subgroup of G in which H is normal. $N(H)$ is called the *normalizer* of H.
(11) If $M \lhd G$, $N \lhd G$, and $M \cap N = \{e\}$, show that for $m \in$M, $n \in N$, $mn = nm$.
(12) If $N \lhd G$ and $H \leq G$, show that $H \cap N \lhd H$.