GROUP THEORY
PROBLEM SET 4
LAGRANGE'S THEOREM

(1) Let G be a group and $H \leq G$.
  (a) Define a binary relation R on G by
$$\forall a, b \in G,\ aRb \text{ if } a^{-1}b \in H.$$
  Show that R is an equivalence relation on G.
  (b) Show that for each $a \in G$, $[a] = aH$, a left coset of H in G.

(2) Let $\{A_i : i \in I\}$ be a partition of a set S i.e. $S = \bigcup_{i \in I} A_i$ and $\forall i \neq j \in I, A_i \cap A_j = \emptyset$. Define a binary relation R on S by
$$\forall a, b \in S,\ aRb \text{ if } a, b \in A_i \text{ for some } i \in I.$$
Show that R is an equivalence relation on S.

(3) Define a binary relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by
$$(a, b) \sim (c, d) \text{ if } ad = bc$$
for any $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Show that $\sim$ is an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. We denote by $\frac{a}{b}$ the equivalence class $[(a, b)]$. For instance,
$$[(1,2)] = \{(1,2), (2,4), (3,6), (4,8), \cdots\}$$
and we denote $[(1,2)]$ by $\frac{1}{2}$. Hence, we see that
$$\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim = \mathbb{Q},$$
the set of all rational numbers. This is how we construct the rational numbers from the integers.

(4) If every right coset of H in G is a left coset of H in G, then show that $aHa^{-1} = H$ for all $a \in G$.