GROUP THEORY
PROBLEM SET 3
SUBGROUPS

(1) Let G be a group. Then show that the center of G, $Z(G) = \{z \in G : zx = xz \ \forall x \in G\}$ is a subgroup of G.

(2) Let G be a group and $H \leq G$. Show that $\forall a \in G$, $a^{-1}Ha = \{a^{-1}ha : h \in H\}$ is also a subgroup of G.

(3) Show that every subgroup of $(\mathbb{Z}, +)$ is of the form $n\mathbb{Z} = \{nx : x \in \mathbb{Z}\}$, $n \geq 0$.

(4) Show that $b\mathbb{Z} \subset a\mathbb{Z} \iff a|b$.

(5) Find all subgroups of $(\mathbb{Z}_4, +)$.

(6) Find all subgroups of $(\mathbb{Z}_6, +)$.

(7) If A, B are subgroups of G, show that $A \cap B$ is a subgroup of G.

(8) Prove that $Z(G) = \bigcap_{a \in G} C(a)$.

(9) Show that $a \in Z(G)$ if and only if $C(a) = G$.

(10) If G is an abelian group and if $H = \{a \in G : a^2 = e\}$, show that $H \leq G$.

(11) Let G be a group and $H \leq G$. Let $Hx = \{hx : h \in H\}$. Show that $\forall a, b \in G$, either $Ha = Hb$ or $Ha \cap Hb = \emptyset$.

The result in this problem will be important in studying Lagrange's Theorem and normal subgroups.