GROUP THEORY
PROBEM SET 2
EXAMPLES OF GROUPS

(1) Determine if each set G with the operation $*$ indicated form a group. If not, point out which of the group axioms fail(s).
  (a) $G = \mathbb{Z}$, $a * b = a - b$.
  (b) $G = \mathbb{Z}$, $a * b = a + b + ab$.
  (c) $G = \mathbb{N} \cup \{0\}$, $a * b = a + b$. Here $\mathbb{N}$ denotes the set of all natural numbers $\{1, 2, 3, \cdots\}$.
  (d) $G =$ set of all rational numbers $\neq -1$, $a * b = a + b + ab$.
  (e) $G =$ set of all rational numbers with denominator divisible by 5 (written so that numerator and denominator are relatively prime), $a * b = a + b$.
  (f) G, a set having more than one element, $a * b = a$ for all $a, b \in G$.

(2) If G is an abelian group, prove that $(a * b)^n = a^n * b^n\ \forall n \in \mathbb{Z}$.
  Hint: Use the mathematical induction.

(3) If G is any group and $a, b, c \in G$, show that if $a * b = a * c$, then $b = c$, and if $b * a = c * a$, then $b = c$.

(4) If G is a group in which $a^2 = e\ \forall a \in G$, show that G is abelian.

(5) Show that any group of order 4 or less is abelian.

(6) Express $(a * b)^{-1}$ in terms of a^{-1} and b^{-1}.

(7) In any group G, prove that $(a^{-1})^{-1} = a\ \forall a \in G$.

(8) Show that a group of order 5 must be abelian.

(9) If G is any group, show that
  (a) e is unique.
  (b) Given $a \in G$, $a^{-1} \in G$ is unique.

(10) Complete the following multiplication table for D_3.

$\cdot$	e	ρ	ρ^2	μ_1	μ_2	μ_3
e						
ρ						
ρ^2						
μ_1						
μ_2						
μ_3						

(11) Complete the following multiplication table for D_4.

$\cdot$	e	ρ	ρ^2	ρ^3	μ_1	μ_2	δ_1	δ_2
e								
ρ								
ρ^2								
ρ^3								
μ_1								
μ_2								
δ_1								
δ_2								