FUNCTIONAL ANALYSIS
PROBLEM SET 6

(1) Show that the functional defined on $\mathscr{C}[a,b]$ by
$$f(x) = \int_a^b x(t)y_0(t)dt,$$
where $y_0 \in \mathscr{C}[a,b]$ is fixed, is linear and bounded.

(2) The null space $\mathscr{N}(M^*)$ of a set $M^* \subset X^*$ is defined to be the set of all $x \in X$ such that $f(x) = 0$ for all $f \in M^*$. Show that $\mathscr{N}(M^*)$ is a vector space.

(3) Let $f \neq 0$ be any linear functional on a vector space X and x_0 any fixed element of $X - \mathscr{N}(f)$, where $\mathscr{N}(f)$ is the null space of f. Show that any $x \in X$ has a unique representation $x = \alpha x_0 + y$, where $y \in \mathscr{N}(f)$.

(4) Determine the null space of the operator $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ represented by
$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{bmatrix}.$$

(5) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by
$$(\xi_1, \xi_2, \xi_3) \longmapsto (\xi_1, \xi_2, -\xi_1 - \xi_2).$$
Find $\mathscr{R}(T)$, $\mathscr{N}(T)$ and a matrix which represents T.

(6) Find the dual basis of the basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ for $\mathbb{R}^3$.

(7) Let $\{f_1, f_2, f_3\}$ be the dual basis of $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$, where $e_1 = (1,1,1)$, $e_2 = (1,1,-1)$, $e_3 = (1,-1,-1)$. Find $f_1(x)$, $f_2(x)$, $f_3(x)$, where $x = (1.0, 0)$.

(8) Let X and Y be normed spaces and $T_n : X \longrightarrow Y$ ($n = 1, 2, \cdots$) bounded linear operators. Show that convergence $T_n \to T$ implies that for every $\epsilon > 0$ there is an N such that for all $n > N$ and all x in any given closed ball we have $||T_n x - Tx|| < \epsilon$.