FUNCTIONAL ANALYSIS
PROBLEM SET 5

(1) Let X be the vector space of all complex 2×2 matrices and define $T : X \longrightarrow X$ by $Tx = bx$, where $b \in X$ is fixed and bx denotes the usual matrix multiplication. Show that T is linear. Under what condition does T^{-1} exist?

(2) Let $T : \mathscr{D}(T) \longrightarrow Y$ be a linear operator whose inverse exists. If $\{x_1, \cdots, x_n\}$ is a linearly independent set in $\mathscr{D}(T)$, show that the set $\{Tx_1, \cdots, Tx_n\}$ is linearly independent.

(3) Let $T : X \longrightarrow Y$ be a linear operator and $\dim X = \dim Y = n < \infty$. Show that $\mathscr{R}(T) = Y$ if and only if T^{-1} exists.

(4) Let X and Y be normed spaces. Show that a linear operator $T : X \longrightarrow Y$ is bounded if and only if T maps bounded sets in X into bounded sets in Y.

(5) If $T \neq O$ is a bounded linear operator, show that for any $x \in \mathscr{D}(T)$ such that $||x|| < 1$ we have the strict inequality $||Tx|| < ||T||$.

(6) Define an operator $T : \ell^\infty \longrightarrow \ell^\infty$ by

$$T(\xi_j) = \left(\frac{\xi_j}{j} \right)$$

for each $(\xi_j) \in \ell^\infty$. Show that T is linear and bounded.

(7) Let T be a bounded linear operator from a normed space X onto a normed space Y. Suppose that there is a positive b such that

$$||Tx|| \geq b||x||$$

for all $x \in X$. Show that $T^{-1} : Y \longrightarrow X$ exists and bounded.