FUNCTIONAL ANALYSIS
PROBLEM SET 4

(1) Let V be the vector space of all bounded or unbounded sequences of complex numbers.
   (a) Define $d : V \times V \longrightarrow \mathbb{R}^+ \cup \{0\}$ by

   $$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|}.$$

   Show that d is a metric on V.
   (b) Show that d is not translation invariant.

(2) Show that in a normed space X, vector addition and multiplication by scalars are continucous operations with respect to the norm. That is, the mappings defined by $(x,y) \longmapsto x+y$ and $(\alpha,x) \longmapsto \alpha x$ are continuous.

(3) Show that $x_n \to x$ and $y_n \to y$ implies $x_n + y_n \to x + y$. Show that $\alpha_n \to \alpha$ and $x_n \to x$ implies $\alpha_n x_n \to \alpha x$.

(4) Show that the closure $\bar{Y}$ of a subspace Y of a normed space X is again a vector subspace.

(5) In a normed space, convergence of a series implies absolute convergence of that series. But the converse need not be true. In fact, we see, throughout the questions (a)-(c), that in a normed space X absolute convergence of a series implies convergence of that series if and only if X is complete.
   (a) Show that in a normed space absolute convergence of a series does not necessarily imply convergence of that series.
   (b) If in a normed space X, absolute convergence of any series always implies convergence of that series, show that X is complete.
   (c) Show that in a Banach space, an absolutely convergent series is convergent.

(6) Show that (e_n), where $e_n = (\delta_{nj})$, is a Schauder basis for ℓ^p, where $1 \leq p < \infty$.