FUNCTIONAL ANALYSIS
PROBLEM SET 3

(1) If $f \geq 0$, f is continuous on $[a,b]$ and $\int_a^b f(x)dx = 0$, then $f(x) = 0$ for all $x \in [a,b]$.
(2) Prove or disprove the following statement: a sequence (x_n) is a Cauchy sequence if and only if given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{n+1} - x_n| < \epsilon$ for all $n \geq N$.
(3) If a sequence (x_n) in a metric space X is convergent and has a limit x, show that every subsequence (x_{n_k}) of (x_n) is convergent and has the same limit x.
(4) If (x_n) is Cauchy and has a convergent subsequence, say, $x_{n_k} \to x$, show that (x_n) is convergent with the limit x.
(5) Show that $x_n \to x$ if and only if for every neighborhood $U(x)$ there exists $N \in \mathbb{N}$ such that $x_n \in U(x)$ for all $n \geq N$.
(6) Show that a Cauchy sequence is bounded.
(7) Is boundedness of a sequence in a metric space sufficient for the sequence to be Cauchy? Convergent?
(8) If $x_n \to x$ in a metric space (X,d), then show that $\lim_{n\to\infty} d(x_n, y) = d(x, y)$ for any $y \in X$.
(9) If (x_n) and (y_n) are Cauchy sequences in a metric space (X,d), show that (a_n), where $a_n = d(x_n, y_n)$, converges.
(10) If a Cauchy sequence (x_n) in a metric space has a limit point x, show that $x_n \to x$.