Discrete Mathematics
Problem Set 4
Strong Induction

1. Let $a_1 = 0$, $a_2 = 6$ and $a_3 = 9$. For $n > 3$, $a_n = a_{n-1} + a_{n-3}$. Show that for all n, a_n is divisible by 3.

2. Show that for all $n \geq 8$, there are integers a and b for which $n = 3a + 5b$.

   Hint: Use ordinary induction on n. In fact, it is true that for all integer n, there are integers a and b such that $n = 3a + 5b$.

3. Let S be the sequence $a_1, a_2, a_3, \cdots$ where $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Use strong induction to prove that $a_n < 2^n$ for any positive integer n.

4. Show that the prime decomposition is unique, i.e. if

   $$n = \prod_{i=1}^{k} p_i^{e_i} = \prod_{i=1}^{\ell} q_i^{f_i}$$

   where the p_i are increasing and are all prime, the q_i are increasing and are all prime, and the e_i and f_i are positive integers, then $k = \ell$, $p_i = q_i$ for each i, and $e_i = f_i$ for each i.

5. Use the method presented in the proof of the Fundamental Theorem of Arithmetic (`https://sunglee.us/mathphysarchive/?p=3581`) to find the prime decomposition of 100. Show the values of S, p, and q at each stage.