Discrete Mathematics
Problem Set 12
Equivalence Relations

1. Let R be an equivalence relation on a set A. Prove, only by using the definition of an equivalence relation, that if $(x,y) \in R$, then $[x] = [y]$.

2. Let $\{A_i\}_{i \in I}$ be a family of subsets of a set A i.e. $A_i \subseteq A\ \forall i \in I$. Then show that
$$\bigcup_{i \in I} A_i \subseteq A$$

3. Let R be an equivalence relation on a set A. Then show that
$$A = \bigcup_{x \in A} [x]$$

4. Let $\{A_i\}_{i \in I}$ be a partition of a set A. Define a relation R on A as follows:
$$\forall x, y \in A,\ (x,y) \in R \Longleftrightarrow x, y \in A_i \text{ for some } i \in I$$
Then prove that R is an equivalence relation on A and that the R-equivalence classes coincide with the A_i's.

5. (a) Let $f : A \longrightarrow B$ be a function. Then prove that the kernel of f
$$\ker f = \{(x,y) \in A \times A : f(x) = f(y)\}$$
is an equivalence relation on A.

   (b) $\forall x \in A$, prove that $[x] = f^{-1}(z)$ where $f(x) = z$.

(c) Suppose that $f : A \longrightarrow B$ is surjective (onto). Define a map $\psi : A/\ker f \longrightarrow B$ by

$$\forall [x] \in A/R,\ \psi([x]) = f(x)$$

Then prove that ψ is a bijective and that $\gamma = \psi \circ f$, where $\gamma : A \longrightarrow A/\ker f$ is the canonical map.

6. (a) Let $\mathbb{Z}$ be the set of all integers. Define a relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by

$$\forall (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}),\ (a,b) \sim (c,d) \Longleftrightarrow ad = bc$$

Then prove that $\sim$ is an equivalence relation.

(b) $\forall (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, define

$$(a,b) + (c,d) = (ad + bc, bd)$$
$$(a,b) \cdot (c,d) = (ac, bd)$$

Prove that the operations $+$ and $\cdot$ are well-defined and that the equivalence relation $\sim$ preserves these operations, i.e. if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ then

$$(a,b) + (c,d) \sim (a',b') + (c',d')$$
$$(a,b) \cdot (c,d) \sim (a',b') \cdot (c',d')$$

Therefore, the addition $+$ and the multiplication $\cdot$ defined on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})/\sim$ by

$$[(a,b)] + [(c,d)] := [(a,b) + (c,d)]$$
$$[(a,b)] \cdot [(c,d)] = [(a,b) \cdot (c,d)]$$

$\forall [(a,b)], [(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})/\sim$ are well-defined.

7. (a) Let $\mathbb{Z}$ be the set of all integers. Define a relation $\equiv \mod n$ on $\mathbb{Z}$ by

$$\forall p,q \in \mathbb{Z},\ p \equiv q \bmod n \Longleftrightarrow n|(p-q)$$

Then prove that $\equiv \mod n$ is an equivalence relation.

(b) If $a \equiv b \bmod n$ and $c \equiv d \bmod n$ then prove that

$$a + c \equiv b + d \bmod n$$
$$ac \equiv bd \bmod n$$