# Discrete Mathematics
# Problem Set 11
# Directed Graphs

1. Prove that for any digraph $G$ and any vertices $u$, $v$, $w$ of $G$,

$$d_G(u,w) \le d_G(u,v) + d_G(v,w)$$

   This is called the *triangle inequality* for directed graphs.

2. Consider the directed graph $G$ in Figure 1.

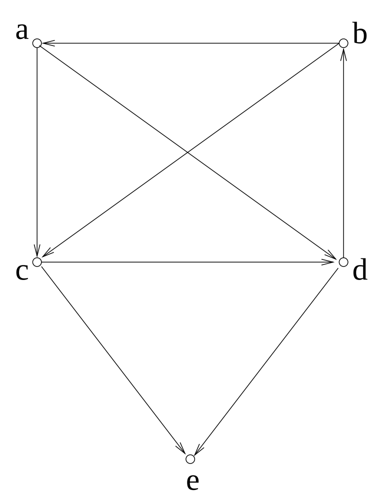


Figure 1: A directed graph $G$

   (a) What is the in-degree and out-degree of each vertex?

   (b) List the cycles of $G$. Do not distinguish two cycles if they trace through the same arcs starting at a different vertex.

(c) Find the distances between each pair of vertices. (There are 25 values.)

(d) What is the length of longest path in $G$? List all paths of that length.

3. Let $G$ be a digraph iwth $V = \{a, b, c, d, e\}$. For which of the following sets of arcs does $G = (V, A)$ contain a cycle?

   (a) $A = \{(a, b), (c, a), (c, b), (d, b)\}$

   (b) $A = \{(a, c), (b, c), (b, d), (c, d), (d, a)\}$

   (c) $A = \{(a, c), (b, d), (c, b), (d, c)\}$

   (d) $A = \{(a, b), (a, d), (b, d), (c, b)\}$

4. Prove that in any digraph, the sum of the in-degrees of all vertices is equal to the sum of the out-degrees of all vertices.

5. (a) Prove that any digraph with $n$ vertices and more than $\frac{n(n-1)}{2}$ arcs contains a cycle.

   (b) What is the minimum value of $m$ that makes the following statement true? Any digraph with $n$ vertices and more than $m$ arcs contains a cycle of length at least 2.

6. Consider the digraph $D = (\mathbb{N} \setminus \{0, 1\}, A)$, where $u \to v \in A$ if and only if $u < v$ and $u|v$.

   (a) Draw the subgraph induced by the vertex set $\{2, \cdots, 12\}$.

   (b) Which vertices of the infinite digraph $D$ have in-degree 0? 1?

   (c) What are the minimum and maximum out-degrees of any vertex in $D$?

   (d) Prove that $D$ is acyclic.

7. Consider the digraph $(V, A)$, where $V$ is the set of all bit strings and $u \to v \in A$ if and only if $v = u0$ or $v = u1$.

   (a) What are the in-degrees and out-degrees of the vertices?

   (b) Under what circumstances is vertex $v$ reachable from vertex $u$?