FUNDAMENTAL THEOREM OF CALCULUS

Let $f(x)$ be a continuous function on $[a, b]$.

Theorem 1. *$F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$.*

Proof. Let $c \in [a, b]$ and let $\epsilon > 0$ be given.

$$\begin{aligned} |F(x) - F(c)| &= \left| \int_a^x f(t)dt - \int_a^c f(t)dt \right| \\ &= \left| \int_a^x f(t)dt + \int_c^a f(t)dt \right| \\ &= \left| \int_c^x f(t)dt \right|. \end{aligned}$$

By the MVT, there exists $d_x \in [c, x]$ such that

$$f(d_x)(x - c) = \int_c^x f(t)dt.$$

Choose $\delta = \frac{\epsilon}{|f(d_x)|+1}$. Then whenever $|x - c| < \delta$,

$$\begin{aligned} |F(x) - F(c)| &= |f(d_x)(x - c)| \\ &= |f(d_x)||x - c| \\ &< (|f(d_x)| + 1)|x - c| \\ &=< (|f(d_x)| + 1)\frac{\epsilon}{|f(d_x)| + 1} \\ &= \epsilon. \end{aligned}$$

Therefore, $F(x)$ is continuous on $[a, b]$. □

Theorem 2. *$F(x)$ is differentiable on (a, b).*

Proof.

$$\begin{aligned}\frac{d}{dx}F(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\\ &= \lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}\\ &= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}.\end{aligned}$$

By the MVT, there exists $c \in [x, x+h]$ such that

$$f(c_h)h = \int_x^{x+h} f(t)dt.$$

Hence,

$$\frac{d}{dx}F(x) = \lim_{h\to 0} f(c_h) = f(x).$$

□

Theorem 3. *If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let $G(x) := \int_a^x f(t)dt$. Then by the FTC Part I, $G(x)$ is continuous on $[a,b]$, differentiable on (a,b), and $G'(x) = f(x)$. So $G(x)$ is an antiderivative of $f(x)$ on (a,b). If $F(x)$ is any antiderivative of $f(x)$, then $F(x) = G(x) + C$ for some constant C and for $a < x < b$. Since both F and G are continuous on $[a,b]$,

$$\begin{aligned}F(b) &= \lim_{x\to b-} F(x)\\ &= \lim_{x\to b-}(G(x)+C)\\ &= G(b)+C\end{aligned}$$

and

$$\begin{aligned}F(a) &= \lim_{x\to a+} F(x)\\ &= G(a)+C.\end{aligned}$$

Therefore,

$$\begin{aligned} F(b) - F(a) &= (G(b) + C) - (G(a) + C) \\ &= G(b) - G(a) \\ &= \int_a^b f(x)dx - \int_a^a f(x)dx \\ &= \int_a^b f(x)dx. \end{aligned}$$

□