Differential Equations
Problem Set 7
Second-Order Linear Equations and Linear Algebra

1. For each of the following linear systems, find the general solution.

   (a) $\frac{dX}{dt} = \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} X$

   (b) $\frac{dX}{dt} = \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} X$

2. Prove the following theorem: *Suppose that $X(t)$ is a complex-valued solution to a linear system $\frac{dX}{dt} = AX$ where the 2×2 matrix A has all real entries. Then the real part $X_{\text{re}}(t)$ and the imaginary part X_{im} of $X(t)$ are both solutions of the linear system.*

   Hint: Write $X(t) = X_{\text{re}}(t) + iX_{\text{im}}(t)$.

3. For each of the following linear systems, use the theorem in #2 to find the *real* general solution. Note that the matrix in each system has complex eigenvalues.

   (a) $\frac{dX}{dt} = \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix} X$

   (b) $\frac{dX}{dt} = \begin{pmatrix} 1 & 4 \\ -3 & 2 \end{pmatrix} X$

4. Suppose that $\frac{dX}{dt} = AX$ is a linear system in which the 2×2 matrix A has a repeated eigenvalue λ. Let v_1 be an eigenvector associated with λ. Then $X_1(t) = e^{\lambda t} v_1$ is a solution of the system. Show that

$$X_2(t) = e^{\lambda t}(t v_1 + v_2),$$

where v_2 satisfies $(A - \lambda I)v_2 = v_1$, is also a solution of the linear system. Since $X_1(t)$ and $X_2(t)$ are linearly independent, the general solution $X(t)$ is given by

$$X(t) = A_1 e^{\lambda t} v_1 + A_2 e^{\lambda t}(t v_1 + v_2)$$

5. For each of the following linear systems, use # 4 to find the general solution. Note that the matrix in each system has a repeated eigenvalue.

   (a) $\frac{dX}{dt} = \begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix} X$

   (b) $\frac{dX}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} X$