Differential Equations
Problem Set 15
The Laplace Transform: Differential Equations with Variable Coefficients

1. Solve the following equations.

   (a) $\ddot{X}(t)+t\dot{X}(t)-2X(t)=1$, $X(0)=\dot{X}(0)=0$.

   (b) $t\ddot{X}(t)+(2t+3)\dot{X}(t)+(t+3)X(t)=e^{-t}$

   (c) $t\ddot{X}(t)+(1-a-t)\dot{X}(t)+aX(t)=t-1$, $X(0)=0$, $a>0$ and $a\neq 1$.

   (d) $t\ddot{X}(t)-(2t+1)\dot{X}(t)+(t+1)X(t)=0$, $X(0)=0$.

2. The Laplace transform of Bessel's equation

$$t^2\ddot{X}(t)+t\dot{X}(t)+(t^2-n^2)X(t)=0$$

   is

$$(s^2+1)x''(s)+3sx'(s)+(1-n^2)x(s)=0$$

   This equation is unfortunately not any easier to solve than Bessel's equation. Let $X(t)=t^{-n}Y(t)$.

   (a) Show that $Y(t)$ satisfies the equation

$$t\ddot{Y}(t)+(1-2n)\dot{Y}(t)+tY(t)=0$$

   (b) Solve the equation in part (a) with condition $Y(0)=0$ and derive the solution $X(t)=CJ_n(t)$ of Bessel's equation where $J_n(t)$, $n=0,1,2,\cdots$ is Bessel fuction of the first kind

$$J_n(t)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\frac{t}{2}\right)^{n+2k}$$