Differential Equations
Problem Set 1
First-Order Differential Equations: Separable, Linear

1. Solve each of the following differential equation.

   (a) $\frac{dy}{dt} = 2y + 1$

   (b) $\frac{dy}{dt} = \frac{t^2+1}{y^4+3y}$

   (c) $\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$

   (d) $\frac{dy}{dx} = \frac{x^2(1+y)}{y^2(1-x)}$

   (e) $\sqrt{1-y^2}dx + \sqrt{1-x^2}dy = 0$

   (f) $\frac{dy}{dt} = -\frac{y}{t} + 2$

   (g) $\frac{dy}{dt} = -2y + \sin t$

   (h) $y' - y = e^{2x}$

   (i) $\frac{dy}{dx} + 2y\tan x = \sin x$

2. Solve each of the following initial value problems.

   (a) $\frac{dy}{dt} = ty^2 + 2y^2,\ y(0) = 1$

   (b) $y' + y\tan x = \sin 2x,\ y(0) = 1$

   (c) $\frac{dy}{dt} = \frac{2y}{t} + 2t^2,\ y(-2) = 4$

   (d) $x\frac{dy}{dx} + y = x\ln x,\ y(1) = 0$

   (e) (RL circuit) $L\frac{dI(t)}{dt} + RI(t) = V_0 \sin\omega t,\ I(0) = 0$

3. (Mathematical Economics: Maximaizing Profit) Let $u(t)$ denote revenue of a businees at t. Assume that the growth rate of revenue is proportional to revenue. This is a reasonale assumption because more money can be invested in the business as revenue grows. Consequently we obtain the differential equation

$$\frac{du}{dt} = au,\ u(0) = u_0 \tag{1}$$

whose solution is $u(t) = u_0 e^{at}$ i.e. the revenue growth is exponential. However this model is not realistic as there is no return of profit. The most important purpose of a business is of course growing profit. Hence the model (1) is modified to

$$\frac{du}{dt} = au - \frac{dp}{dt},\ u(0) = u_0 \tag{2}$$

where $p = p(t)$ is profit at t. It is also reasonable to assume that the growth rate of profit is proportional to revenue. Consequently we obtain the differential equation

$$\frac{dp}{dt} = ku,\ p(0) = 0 \tag{3}$$

$p(0) = 0$ happens when, for instance, the entire intial revenue is used to cover the initial investment and other cost of business. The revenue model in (2) then turns into

$$\frac{du}{dt} = a(1-k)u,\ u(0) = u_0 \tag{4}$$

whose solution is

$$u(t) = u_0 e^{a(1-k)t} \tag{5}$$

(a) Using (5) show that

$$p(t) = \frac{ku_0}{a(1-k)}[e^{a(1-k)t} - 1] \tag{6}$$

It is an interesting question to consider for What values of k the business must choose in order to maximize $p(t)$ in (6). If $k = 0$, the revenue grows expeonentially as we have seen but thre is no profit. If $k = 1$, there is no revenue growth as all revenue returns to profit.

(b) For large t, $p(t)$ in (6) can be approximated as

$$p(t) \approx \frac{ku_0 e^{a(1-k)t}}{a(1-k)}$$

What would be the value of k that maximizes $p(t)$ for fixed t? Will there be only one such k value?

(c) Show that for small t $p(t)$ in (6) can be approximated as

$$p(t) \approx ku_0 \left[t + \frac{a(1-k)}{2} t^2 \right]$$

What would be the value of k that maximizes $p(t)$ for fixed t? Also show that $k = 1$ until t reaches a certain point in time.

(Torricelli's Law) Suppose that a water tank has a hole with area a at its bottom and that water is draining from the hole. Denote by $y(t)$ the depth (in feet) of water in the tank at time t (in seconds) and by $V(t)$ the volume of water (in cubic feet) in the tank. Then the velocity of the stream of water exiting through the hole is

$$v = \sqrt{2gy} \ (g \approx 32\text{ft/sec}^2).$$

This is *Torricelli's law of draining*. The amount of water that leaves through the bottom hole during a short time interval dt amounts to a cylinder with base area a and height vdt. Hence, the resulting change dV in the volume of water in the tank is given by

$$dV = -avdt = -a\sqrt{2gy}dt. \tag{7}$$

On the other hand if $A(y)$ denotes the horizontal cross-sectional area of the tank at height y above the hole, then

$$dV = A(y)dy. \tag{8}$$

Comparing equations (7) and (8), we see that $y(t)$ satisfies the differential equation

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}. \tag{9}$$

4. A hemispherical tank has top radius 4 ft and, at time $t = 0$,is full of water. At that moment a circular hole of diameter 1 in. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank? Use the differential equation (9).

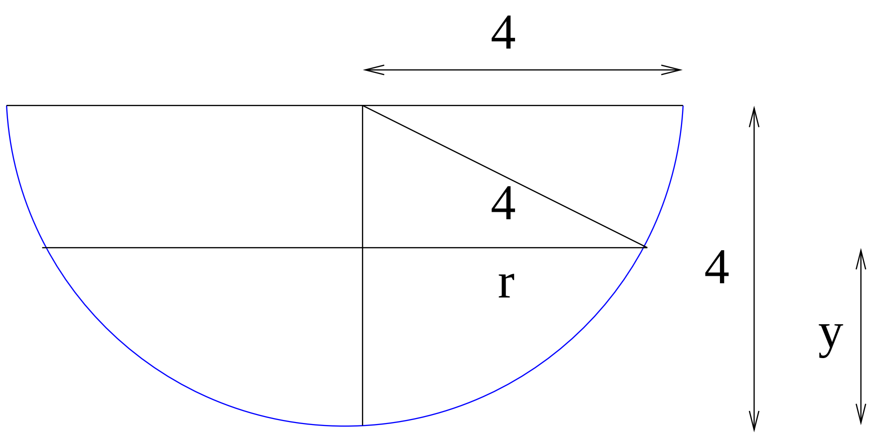

Figure 1: Hemispherical water tank

5. (Which is Faster, Going Up or Coming Down?) Suppose you throw a ball into the air. Do you think it takes longer to reach its maximum height or to fall back to Earth from its maximum height?

   (a) A ball with mass m is projected vertically upward from Earth's surface with an initial velocity v_0. Assume the forces acting on the ball are the force of gravity and a retarding force of air resistence with direction opposite to the direction of motion and with the magnitude $p|v(t)|$, where p is a positive constant and $v(t)$ is the velocity of the ball at time t. In both the ascent and the descent, the total force acting on the ball is $-pv - mg$. (During ascent, $v(t)$ is positive and the resistance acts downward; during descent, $v(t)$ is negativeand the resistence acts upward.) So, by Newton's Second Law, the equation of motion is
   $$m\frac{dv}{dt} = -pv - mg.$$
   Solve this differential equation to show that the velocity is
   $$v(t) = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}.$$

(b) Show that the height of the ball, until it hits the ground, is

$$h(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}(1 - e^{-pt/m}) - \frac{mgt}{p}.$$

(c) Let t_1 be the time that the ball takes to reach its maximum height. Show that

$$t_1 = \frac{m}{p}\ln\left(\frac{mg + pv_0}{mg}\right).$$

Find this time for a ball with mass 1 kg and initial velocity 20 m/sec. Assume the air resistence is $\frac{1}{10}$ of the speed.

(d) Let t_2 be the time at which the ball falls back to Earth. For the particular ball in part (c), estimate t_2 by using a graph of the height function $h(t)$. Which is faster, going up or coming down?

(e) In general, it's not easy to find t_2 because it's impossible to solve the equation $h(t) = 0$ explicitly. We can, however, use an indirect method to determine whether ascent or descent is faster; we determine whether $h(2t_1)$ is positive or negative. Show that

$$h(2t_1) = \frac{m^2 g}{p^2}\left(x - \frac{1}{x} - 2\ln x\right),$$

where $x = e^{pt_1/m}$. Then show that $x > 1$ and the function

$$f(x) = x - \frac{1}{x} - 2\ln x$$

is increasing for $x > 1$. Use this result to decide whether $h(2t_1)$ is positive or negative. What can you conclude? Is ascent or descent faster?