FUNCTIONS OF A COMPLEX VARIABLE
PROBLEM SET 4: TAYLOR SERIES, LAURENT SERIES

(1) Show that

$$e^z = e + e\sum_{n=1}^{\infty}\frac{(z-1)^n}{n!}$$

when $|z| < \infty$.

(2) Using the definitions $\sinh z = \frac{e^z - e^{-z}}{2}$ and $\cosh z = \frac{e^z + e^{-z}}{2}$, find the Maclaurin series representations for $\sinh z$ and $\cosh z$.

(3) Using

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

for $|z| < 1$, find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4 + 9}.$$

(4) Prove that when $z \neq 0$,

$$\frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots .$$

(5) Obtain the Maclaurin series representations

$$z\cosh(z^2) = z + \sum_{n=1}^{\infty}\frac{1}{(2n)!}z^{4n+1},$$

when $|z| < \infty$.

(6) Obtain the Laurent series expansions

(a) $\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{1}{(2n+1)!}z^{2n-1}$, when $0 < |z| < \infty$;

(b) $z^3\cosh\left(\frac{1}{z}\right) = \frac{z}{2} + z^3 + \sum_{n=1}^{\infty}\frac{1}{(2n+2)!}\frac{1}{z^{2n-1}}$ $(0 < |z| < \infty)$.

(7) Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}.$$

(8) Give two Laurent series expansions in powers of z for the functions

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which those expansions are valid.

(9) Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$.

(10) Derive the Laurent series expansion

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left[\sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right] \quad (0 < |z+1| < \infty)$$

(11) Find a Laurent series representation for the function

$$f(z) = \frac{1}{1+z}$$

when $1 < |z| < \infty$.

(12) Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$