FUNCTIONS OF A COMPLEX VARIABLE
PROBLEM SET 3: THE LINE INTEGRAL, CAUCHY'S INTEGRAL THEOREM, CAUCHY'S INTEGRAL FORMULA

(1) Use parametric representation for the contour C to evaluate $\int_C f(z)dz$ where $f(z) = \frac{z+2}{z}$ and C is
  (a) the semicircle $z = 2e^{i\theta}$ $(0 \leq \theta \leq \pi)$;
  (b) the semicircle $z = 2e^{i\theta}$ $(\pi \leq \theta \leq 2\pi)$;
  (c) the circle $z = 2e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$.

(2) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Show without evaluating the integral that

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3}.$$

(3) Let C_R denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \leq \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Use this inequality to show that

$$\lim_{R \to \infty} \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz = 0.$$

(4) Let C denote the positively oriented boundary of the square whose sides like along the lintes $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:
  (a) $\int_C \frac{e^{-z}}{z - \frac{\pi i}{2}} dz$
  (b) $\int_C \frac{\cos z}{z(z^2+8)} dz$
  (c) $\int_C \frac{z}{2z+1} dz$
  (d) $\int_C \frac{\cosh z}{z^4} dz$

(5) Let C be the unit circle $z = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$. First show that, for any real constant a

$$\int_C \frac{e^{az}}{z} dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta} \cos(a \sin\theta) d\theta = \pi.$$