FUNCTIONS OF A COMPLEX VARIABLE
PROBLEM SET 2: CAUCHY-RIEMANN CONDITIONS, HARMONIC FUNCTIONS

(1) Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions

$$x = \frac{z + \bar{z}}{2} \text{ and } y = \frac{z - \bar{z}}{2i}$$

to write $f(z)$ in terms of z, and simplify the result.

(2) Show whether or not $f(z) = \text{Re}z = x$ is analytic.

(3) Verify that each of the following functions is entire.
   (a) $f(z) = 3x + y + i(3y - x)$
   (b) $f(z) = \sin x \cosh y + i \cos x \sinh y$

(4) Two-dimensional fluid flow can be described by a complex function $f(z) = u(x, y) + iv(x, y)$. Label the real part $u(x, y)$, the velocity potential and the imaginary part $v(x, y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic, show that
   (a) $\frac{df}{dz} = V_x - iV_y$,
   (b) $\nabla \cdot \mathbf{V} = 0$ (i.e. no source or sinks),
   (c) $\nabla \times \mathbf{V} = 0$ (i.e. irrotational, nonturbulent flow).

(5) Show that each of the following functions is nowhere analytic.
   (a) $f(z) = xy + iy$
   (b) $f(z) = 2xy + i(x^2 - y^2)$

(6) Suppose that a function $f(z) = u(x, y) + iv(x, y)$ and its conjugate $\overline{f(z)} = u(x, y) - iv(x, y)$ are both analytic in a given domain D. Show that $f(z)$ must be constant throughout D.

(7) Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions

$$x = \frac{z + \bar{z}}{2} \text{ and } y = \frac{z - \bar{z}}{2i}$$

to write $f(z)$ in terms of z, simplify the result.

(8) Show whether or not $f(z) = \text{Re}z = x$ is analytic.

(9) Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when
   (a) $u(x, y) = 2x - x^3 + 3xy^2$
   (b) $u(x, y) = \sinh x \sin y$
   (c) $u(x, y) = \frac{y}{x^2+y^2}$

(10) Find the analytic functions

$$f(z) = u(x, y) + iv(x, y)$$

if
   (a) $u(x, y) = x^3 - 3xy^2$,
   (b) $v(x, y) = e^{-y} \sin x$.