FUNCTIONS OF A COMPLEX VARIABLE
PROBLEM SET 1: COMPLEX ALGEBRA

(1) Verify that
 (a) $(\sqrt{2}-i)-i(1-\sqrt{2}i)=-2i$;
 (b) $(2,-3)(-2,1)=(-1,8)$;
 (c) $(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right)=(2,1)$.
(2) Show that
 (a) $\mathrm{Re}(iz)=-\mathrm{Im}z$;
 (b) $\mathrm{Im}(iz)=\mathrm{Re}z$.
(3) Use $i=(0,1)$, $a=(a,0)$ and $b=(b,0)$ where a and b are real numbers to show that

$$a+ib=(a,b).$$

(4) Solve the equation $z^2+z+1=0$ for $z=(x,y)$ by writing

$$(x,y)(x,y)+(x,y)+(1,0)=(0,0)$$

and then solving a pair of simultaneous equations in x and y.
(5) Prove the triangle inequality

$$|z_1+z_2|\le|z_1|+|z_2|.$$

(6) Use the triangle inequality in #1 to show the equality

$$||z_1|-|z_2||\le|z_1+z_2|.$$

(7) Find the principal argument $\mathrm{Arg}z$ when
 (a) $z=\frac{i}{-2-2i}$;
 (b) $z=(\sqrt{3}-i)^6$.
(8) By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that
 (a) $i(1-\sqrt{3}i)(\sqrt{3}+i)=2(1+\sqrt{3}i)$;
 (b) $\frac{5i}{2+i}=1+2i$;
 (c) $(-1+i)^7=-8(1+i)$;

(d) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

(9) Use de Moivre's formula to derive the following trigonometric identities:
 (a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$;
 (b) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

(10) Use

$$1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z} \quad (z \neq 1)$$

to derive Lagrange's trigonometric identity:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[\frac{1}{2}(2n+1)\theta\right]}{2\sin\left(\frac{\theta}{2}\right)},$$

for $0 < \theta < 2\pi$.

(11) Find $\sqrt{i}$.

(12) Let $z = re^{i\theta}$ be a complex number. Then

$$\ln z = \ln r + i(\theta + 2n\pi),$$

where n is any integer. Show that
 (a) $e^{\ln z}$ always equals z.
 (b) $\ln e^z$ does not always equal z.