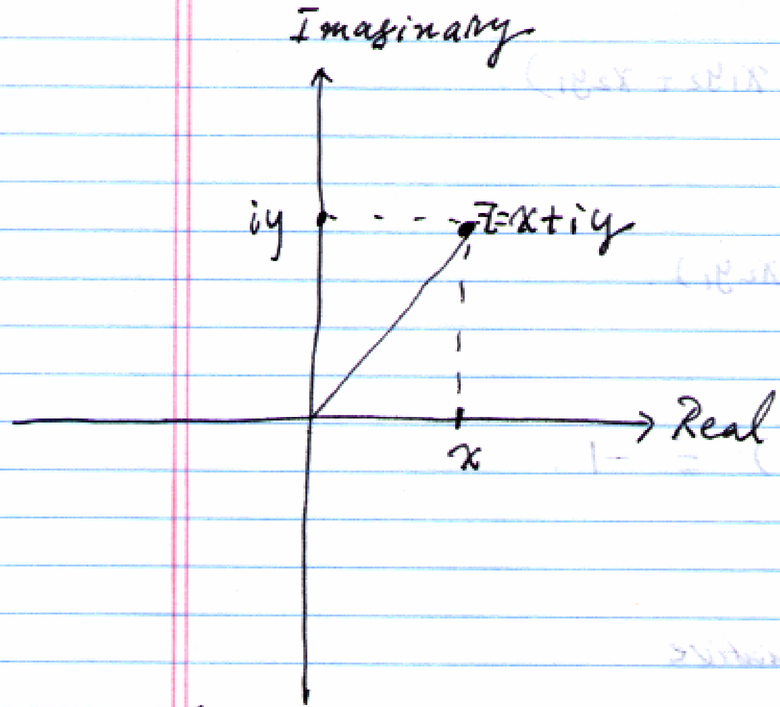
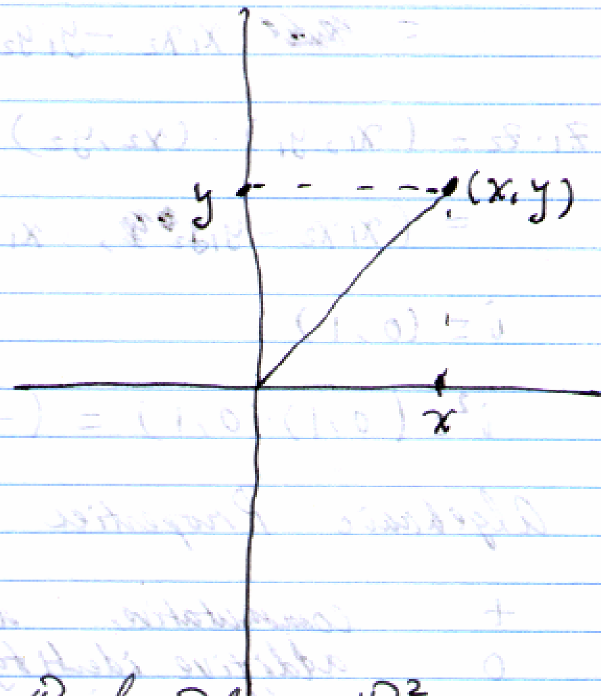
Functions of a Cpx Variable

Complex algebra



Complex Plane $\mathbb{C}$



Real Plane $\mathbb{R}^2$

$$i=\sqrt{-1}, \quad i^2=-1$$

$$z = x+iy = (x,y)$$

$x = \operatorname{Re} z, \quad y = \operatorname{Im} z$

Two complex #s $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$ are equal if and only if $x_1 = x_2$, $y_1=y_2$.

The sum z_1+z_2

$$z_1+z_2 = (x_1+iy_1) + (x_2+iy_2)$$

$$= x_1+x_2 + i(y_1+y_2)$$

$$z_1+z_2 = (x_1,y_1) + (x_2,y_2)$$

$$= (x_1+x_2, \; y_1+y_2)$$

Multiplication $z_1 \cdot z_2$

$$z_1 \cdot z_2 = (x_1 + i y_1)\cdot(x_2 + i y_2)$$
$$= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$$

$$z_1 \cdot z_2 = (x_1, y_1)\cdot(x_2, y_2)$$
$$= (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1).$$

$$i = (0,1)$$

$$i^2 = (0,1)\cdot(0,1) = (-1,0) \overset{\sim}{=} -1.$$

Algebraic Properties

$+$ commutative, associative
0 additive identity
additive inverse

$(\mathbb{C}, +)$ is an abelian group.

$\cdot$ commutative, associative
1 multiplicative identity

For any nonzero cpx #, $\exists$ a multiplicative inverse

Suppose
$z = x + iy \neq 0.$

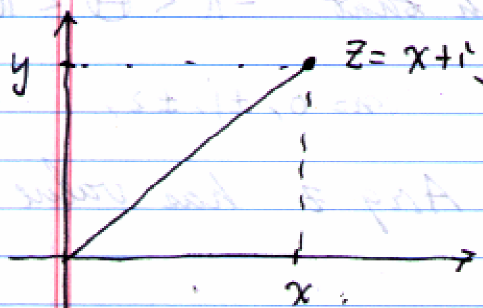
$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

$$z \quad x+iy \ \xleftrightarrow[\text{conjugate}]{} \ \bar{z} = x - iy \qquad = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}.$$

$(\mathbb{C}, +, \cdot)$ is a "field."

Physicists often denote the conjugate of z by z^*.

Functions of a Cpx variable



$|z_1 z_2| = |z_1||z_2|$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

The norm (modulus) (length) of z : $|z| = \sqrt{z \cdot \bar{z}}$

$= \sqrt{x^2+y^2}$.

Eg. $|-3+2i| = \sqrt{(-3+2i)(-3-2i)}$

$= \sqrt{9+4} = \sqrt{13}$.

$|z| = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2}$.

Triangle Inequality

$|z_1+z_2| \le |z_1| + |z_2|$.

Corollary.

$||z_1| - |z_2|| \le |z_1+z_2|$.

Exponential Form

$z = x+iy$ $\quad x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2+y^2}$.

$z = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$.

$\theta = \tan^{-1}\frac{y}{x}$ is called an argument.

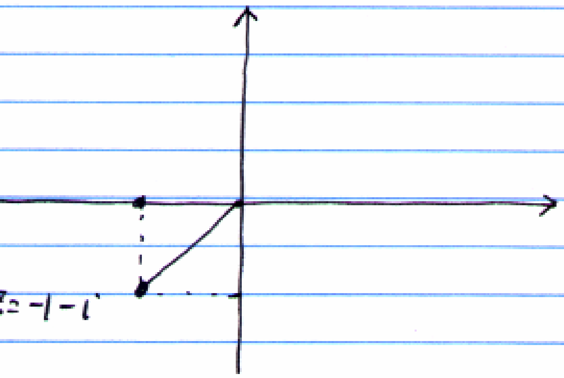
The set of all arguments is denoted by $\arg z$.

The principal value of $\arg z$, denoted by $\operatorname{Arg} z$, is that unique value Θ such that $-\pi < \Theta \leq \pi$.

$$\arg z = \operatorname{Arg} z + 2n\pi, \qquad n = 0, \pm 1, \pm 2, \ldots$$

When z is a negative real #, $\operatorname{Arg} z$ has value π, not $-\pi$.

Eg. $z = -1 - i$.



Principal argument $\Theta = -\frac{3\pi}{4}$.

$$\arg z = \arg(-1-i)$$
$$= -\frac{3\pi}{4} + 2n\pi$$
$$(n = 0, \pm 1, \pm 2, \ldots)$$

$$-\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$

$\arg z$ may not necessarily be represented by the principal argument. For example, $\arg(-1-i)$ may be written as

$$\arg(-1-i) = \frac{5}{4}\pi + 2n\pi, \qquad n = 0, \pm 1, \pm 2, \ldots$$

although $\operatorname{Arg}(-1-i) \neq \frac{5}{4}\pi$.

In exponential form, $-1-i$ may be written in exponential form as

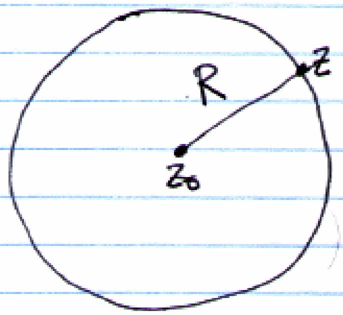
$$-1-i = \sqrt{2}\, \exp\left[i\left(-\frac{3}{4}\pi\right)\right].$$

This is one of infinitely many possibilities for the exponential form of $-1-i$.

$$-1-i = \sqrt{2}\, \exp\left[i\left(-\frac{3\pi}{4} + 2n\pi\right)\right] \quad (n = 0, \pm 1, \pm 2, \ldots)$$

Functions of a Cpx Variable

Equation of a circle centered at z_0 with radius R



$|z - z_0| = R.$

$z = Re^{i\theta}$

$z - z_0 = Re^{i\theta}$ or $z = z_0 + Re^{i\theta}$
$0 \le \theta \le 2\pi.$

Products and Quotients in Exponential Form

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

This identity is not always true when arg is replaced by Arg.

Eg. $\quad z_1 = -1, \quad z_2 = i$

$\text{Arg}(z_1 z_2) = \text{Arg}(-i) = -\frac{\pi}{2}$

$\text{Arg}\, z_1 + \text{Arg}\, z_2 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$

Choose

$\arg(z_1 z_2) = \text{Arg}(z_1 z_2) + 2\pi$

$= -\frac{\pi}{2} + 2\pi = \frac{3\pi}{2} = \arg(z_1) + \arg(z_2).$

Similarly, $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

Eg. Find the principal argument Arg z when

$$z = \frac{-2}{1+\sqrt{3}\,i}.$$

$$\arg z = \arg\left(\frac{-2}{1+\sqrt{3}i}\right)$$

$$= \arg(-2) - \arg(1+\sqrt{3}i)$$

$\operatorname{Arg}(-2) = \pi$, $\operatorname{Arg}(1+\sqrt{3}i) = \frac{\pi}{3}$.

One value of $\arg z$ is $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Since $-\pi < \frac{2\pi}{3} \le \pi$, $\operatorname{Arg}\left(\frac{-2}{1+\sqrt{3}i}\right) = \frac{2}{3}\pi$.

$$z^n = r^n e^{in\theta}, \quad n = 0, \pm 1, \pm 2, \cdots$$

$$(e^{i\theta})^n = e^{in\theta}, \quad n = 0, \pm 1, \pm 2, \cdots$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n = 0, \pm 1, \pm 2, \cdots$$

de Moivre's formula

Eg. Write $(\sqrt{3}+i)^7$ in the form $a+ib$.

$$\sqrt{3}+i = 2e^{i\pi/6}$$

$$(\sqrt{3}+i)^7 = (2e^{i\pi/6})^7 = 2^7 e^{i\frac{7\pi}{6}}$$

$$= (2^6 e^{i\pi})(2e^{i\frac{\pi}{6}})$$

$$= -64(\sqrt{3}+i).$$

Functions of a Cpx Variable

Roots of Cpx Numbers

$$z^n = z_0$$

$z = re^{i\theta}$

$z_0 = r_0 e^{i\theta_0}$

$$r^n e^{in\theta} = r_0 e^{i\theta_0}$$

$$r^n = r_0, \quad n\theta = \theta_0 + 2k\pi$$

$$r = \sqrt[n]{r_0} \qquad \theta = \frac{\theta_0}{n} + \frac{2k}{n}\pi$$

$$k = 0, 1, 2, \cdots, (n-1)$$

$$z = \sqrt[n]{r_0}\, \exp\left[i\frac{(\theta_0 + 2k\pi)}{n}\right], \quad k = 0, 1, 2, \cdots, (n-1).$$

Eg. n-th root of unity

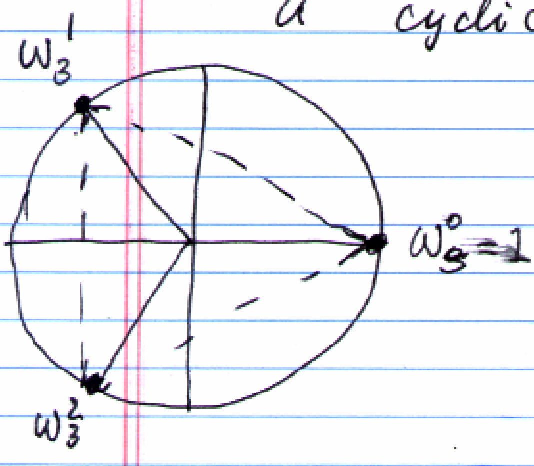
$$z^n = 1.$$

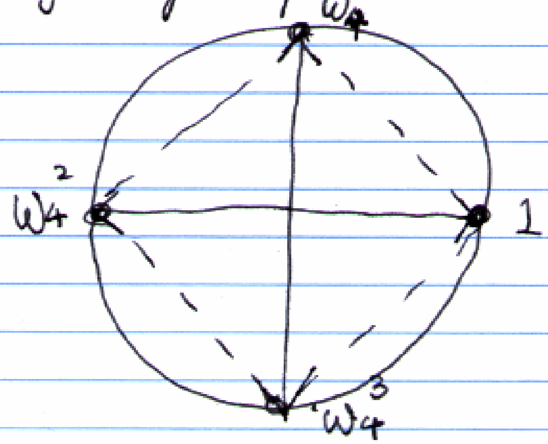
$$1 = r^n e^{in\theta} \qquad r = 1, \quad \theta = \frac{2k\pi}{n}, \quad k = 0, 1, 2, \cdots, (n-1)$$

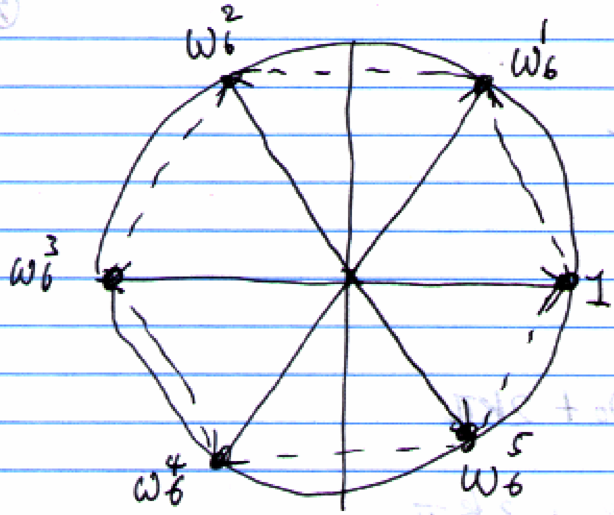
$$\omega_n^k = \exp\left(i\frac{2k\pi}{n}\right)$$

$$1, \omega_n, \omega_n^2, \cdots, \omega_n^{n-1} \cong \mathbb{Z}_n$$

a cyclic group of order n.






ω_6^2
ω_6^1
ω_6^3
1
ω_6^4
ω_6^5