Function of a Complex Variable

Cauchy - Riemann Conditions

Consider a complex-valued function $f(z)$ of a complex variable. The derivative $f'(z)$ would be defined as

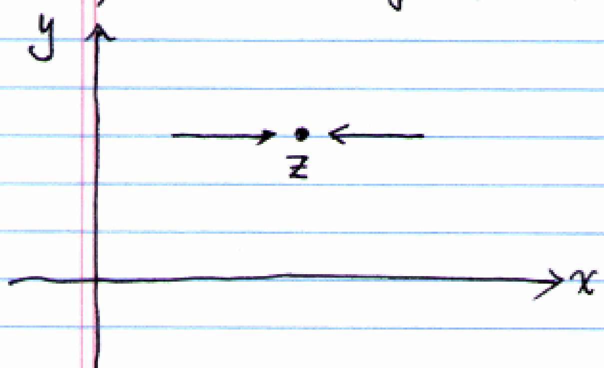
$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

$f'(z)$ is also denoted by $\frac{df}{dz}$.

The limit is independent of the particular approach to the point z. Suppose that $f'(z)$ exists.
Let $z = x + iy$ and $f(z) = u(z) + i v(z)$.

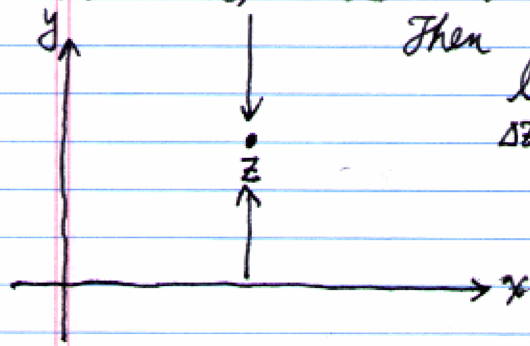
Then $$\frac{\Delta f}{\Delta z} = \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y}.$$

First, let $\Delta y = 0$ and $\Delta x \to 0$.



Then
$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x}$$
$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

Now this time, we let $\Delta x = 0$ and $\Delta y \to 0$.



Then
$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta u + i \Delta v}{i \Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{\Delta v}{\Delta y} - i \frac{\Delta u}{\Delta y}$$
$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

Functions of a Complex Variable

Since $f'(z)$ exists,

the two limits must coincide i.e.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are called the Cauchy-Riemann conditions.
Conversely, let us assume that the C-R conditions are satisfied. Also assume that the partial derivatives of $u(x,y)$ and $v(x,y)$ are continuous.
For small Δx and Δy, we have

$$\Delta f \approx df = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y.$$

Dividing by Δz, we obtain

$$\frac{\Delta f}{\Delta z} \approx \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y}{\Delta x + i\Delta y}$$

$$= \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{\Delta y}{\Delta x}}{1 + i\frac{\Delta y}{\Delta x}}$$

By the C-R,

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x} = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right).$$

So, $\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$

Functions of a Complex Variable

If $f(z)$ is differentiable at $z=z_0$ and in some neighborhood of z_0, we say that $f(z)$ is analytic at $z=z_0$.
If $f(z)$ is analytic everywhere in the complex plane $\mathbb{C}$, it is called an entire function.

Eg. $f(z) = z^2$

$z = x+iy \Rightarrow z^2 = (x+iy)^2 = x^2-y^2+i2xy.$

So, $u(x,y) = x^2-y^2$ and $v(x,y) = 2xy$.

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Hence, $f(z)=z^2$ satisfies the C-R conditions.
Since the partial derivatives are continuous, $f'(z)$ exists and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i2y.$$

Eg. $f(z) = \bar{z} = x - iy$.

$u(x,y) = x$ and $v(x,y) = -y$.

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y},$$ so $f(z)=\bar{z}$ is not analytic.

Remark. A function $f: \mathbb{C} \longrightarrow \mathbb{C}$ may be viewed as $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, a vector-valued function of two real variables.
However, the notion of differentiability is different between the two. For example, let us consider $f(z)=|z|^2$.
It may be viewed as $f(x,y) = x^2+y^2$. As a real-valued function of two real variables, it is differentiable everywhere since $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$ exist everywhere in $\mathbb{R}^2$.
However, as a function of a complex variable, $u(x,y) = x^2+y^2$, $v(x,y) = 0$, so the C-R conditions are not satisfied unless $x=y=0$.

Hence, $f(z) = |z|^2$ is differentiable only at $(0,0)$.
It is not analytic.

Differentiation Formulas

$\frac{dc}{dz} = 0$, where c is a constant complex number.

$\frac{d}{dz} z^n = n z^{n-1}$ if n is a positive integer.

The formula remains valid when n is a negative integer, provided $z \neq 0$.

$$\frac{d[cf(z)]}{dz} = c \frac{df}{dz}$$

$$\frac{d}{dz}[f(z) + g(z)] = \frac{d}{dz} f(z) + \frac{d}{dz} g(z)$$

$$\frac{d}{dz}[f(z) g(z)] = \frac{df(z)}{dz} \cdot g(z) + f(z) \cdot \frac{dg(z)}{dz}$$

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{\frac{df(z)}{dz} \cdot g(z) - f(z) \frac{dg(z)}{dz}}{[g(z)]^2}.$$

If $W = g(\omega)$ and $\omega = g(z)$, then

$$\frac{dW}{dz} = \frac{dg}{d\omega} \cdot \frac{d\omega}{dz} \quad \text{[Chain Rule]}$$

Eg. $\frac{d}{dz}(2z^2 + i)^5 = 20z(2z^2 + i)^4$.

Polar Coordinates

$x = r\cos\theta, \quad y = r\sin\theta$

Functions of a Complex Variable

$f(z) = u(x, y) + i\, v(x, y)$

$f(r, \theta) = u(r, \theta) + i\, v(r, \theta)$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= u_x \cos\theta + u_y \sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= -u_x r \sin\theta + u_y r \cos\theta.$$

Similarly, we obtain

$$v_r = v_x \cos\theta + v_y \sin\theta, \qquad v_\theta = -v_x r \sin\theta + v_y r \cos\theta.$$

Suppose that $f(z)$ satisfies the C-R conditions i.e. $u_x = v_y$ and $u_y = -v_x$.

Then

$$v_r = -u_y \cos\theta + u_x \sin\theta, \qquad v_\theta = u_y r \sin\theta + u_x r \cos\theta.$$

Hence, we get

$$\underbrace{r u_r = v_\theta, \qquad u_\theta = -r v_r.}$$

The Cauchy-Riemann equations in polar coordinates.

$$\boxed{f'(z) = e^{-i\theta}(u_r + i v_r)}$$

Eg. $f(z) = \dfrac{1}{z} = \dfrac{1}{re^{i\theta}} = \dfrac{1}{r} e^{-i\theta} = \dfrac{1}{r}(\cos\theta - i \sin\theta) \qquad (z \neq 0)$

$$u(r, \theta) = \frac{1}{r}\cos\theta, \qquad v(r, \theta) = -\frac{1}{r}\sin\theta$$

$$r u_r = -\frac{\cos\theta}{r} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sin\theta}{r} = -r v_r.$$

$$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = -\frac{1}{z^2}.$$

Harmonic Functions

Suppose that a function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D. Then $f(z)$ satisfies the C-R conditions i.e.

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Differentiating these equations with respect to x, we obtain

$$u_{xx} = v_{yx}, \quad u_{yx} = -v_{xx}.$$

Differentiating ~~the~~ C-R equations with respect to y, we obtain

$$u_{xy} = v_{yy}, \quad u_{yy} = -v_{xy}.$$

By ~~the~~ continuity of the partial derivatives of u and v,

$$u_{xy} = u_{yx} \quad \text{and} \quad v_{xy} = v_{yx}$$

and thereby we get

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0.$$

That is, u and v are harmonic in D.

Eg. The function $f(z) = e^{-y}\sin x - i e^{-y}\cos x$ is entire, so both $e^{-y}\sin x$ and $-e^{-y}\cos x$ are harmonic.

If two functions u, v are harmonic in a domain D and their first-order partial derivatives satisfy the C-R throughout D, v is said to be a harmonic conjugate of u.

Theorem. A function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

Remark. If v is a harmonic conjugate of u in some domain, it is not, in general, true that u is a harmonic conjugate of v there.

Eg. $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.

Since $f(z) = z^2$ is entire, v is a harmonic conjugate of u. However, u cannot be a harmonic conjugate of v since $2xy + i(x^2 - y^2)$ is not analytic anywhere.

Eg. Finding a harmonic conjugate of a harmonic function.

Let $u(x,y) = y^3 - 3x^2 y$, and $v(x,y)$ be a harmonic conjugate of u. Then

$$u_x = v_y, \quad u_y = -v_x.$$

$$v_y = -6xy$$

$$\Rightarrow \quad v(x,y) = -3xy^2 + \phi(x)$$

$$\Rightarrow \quad \begin{array}{ccccc} v_x & = & -3y^2 & + & \phi'(x) \\ \parallel & & & & \parallel \\ -u_y & = & -3y^2 & + & 3x^2 \end{array}$$

$$\Rightarrow \quad \phi'(x) = 3x^2$$

$$\Rightarrow \quad \phi(x) = x^3 + C.$$

Hence, $v(x,y) = -3xy^2 + x^3 + C$.

The corresponding analytic function is

$$f(z) = (y^3 - 3x^2 y) + i(-3xy^2 + x^3 + C).$$