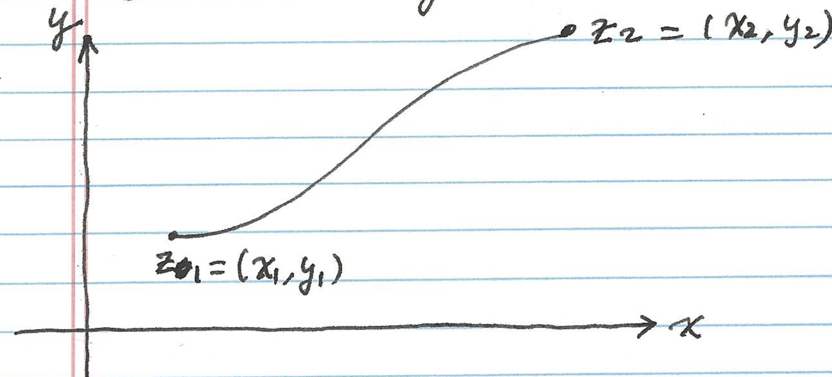
Functions of a Complex Variable

Cauchy's Integral Theorem

Contour Integrals

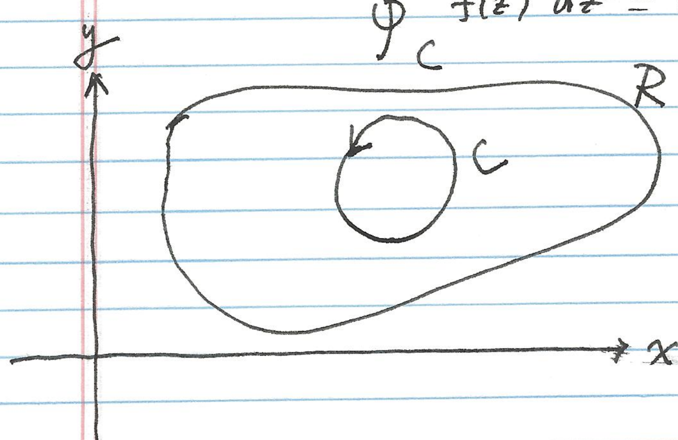


$$\int_{z_1}^{z_2} f(z)\, dz = \int_{(x_1,y_1)}^{(x_2,y_2)} [u(x,y) + i v(x,y)][dx + i dy]$$

$$= \int_{(x_1,y_1)}^{(x_2,y_2)} [u(x,y)\,dx - v(x,y)\,dy] + i \int_{(x_1,y_1)}^{(x_2,y_2)} [v(x,y)\,dx + u(x,y)\,dy]$$

Theorem. If a function $f(z)$ is analytic and its partial derivatives are continuous throughout some simply connected region R, $\forall$ closed path C in R, the line integral of $f(z)$ around C is zero i.e.

$$\oint_C f(z)\, dz = 0.$$



Functions of a Complex Variable

$$f(z) = u(x,y) + i\,v(x,y), \qquad dz = dx + i\,dy$$

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + i\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i \oint_C (v\,dx + u\,dy).$$

Recall Stokes' theorem,

$$\oint_C (V_x\,dx + V_y\,dy) = \int \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) dx\,dy, \text{ where}$$

$V = (V_x, V_y) = V_x \vec{i} + V_y \vec{j}$ is a vector field defined in R such that V_x and V_y have continuous partial derivatives in R.

$$\cancel{\oint \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) dx\,dy \neq \oint}$$

$$\oint_C (u\,dx - v\,dy) = \cancel{\oint_C} \int \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$$

$$= -\int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx\,dy.$$

$$\oint_C (v\,dx + u\,dy) = \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy$$

It then follows from the Cauchy-Riemann conditions that $\oint_C f(z)\,dz = 0.$

Eg. If C is any simply closed contour, in either direction, then

$$\int_C \exp(z^3)\, dz = 0,$$

because $f(z) = \exp(z^3)$ is analytic everywhere and its derivative $f'(z) = 3z^2 \exp(z^3)$ is continuous everywhere.

Multiply - Connected Regions

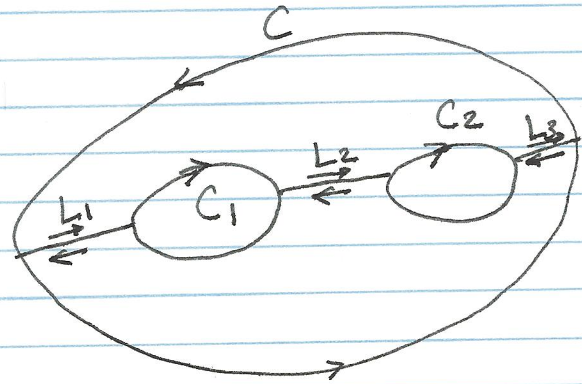
Suppose:

C, a simple closed contour, oriented counterclockwise.

C_k, $k = 1, \ldots, n$ are simple closed contours interior to C, all oriented clockwise, that are disjoint and whose interiors have no points in common.

If a function f is analytic in a ~~region~~ domain (connected and open set) containing C, then

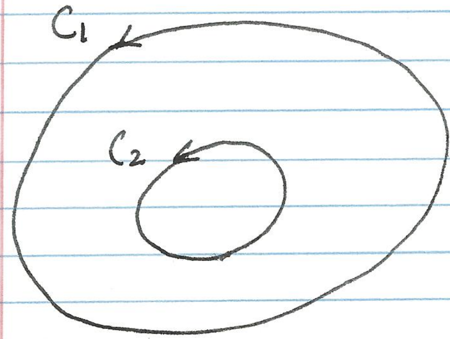
$$\int_C f(z)\, dz + \sum_{k=1}^{n} \int_{C_k} f(z)\, dz = 0.$$



Corollary. Let C_1 and C_2 denote positively oriented simple closed contours, where C_2 is interior to C_1. If a function f is analytic in a domain containing C_1, then

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$$

Functions of a Complex Variable



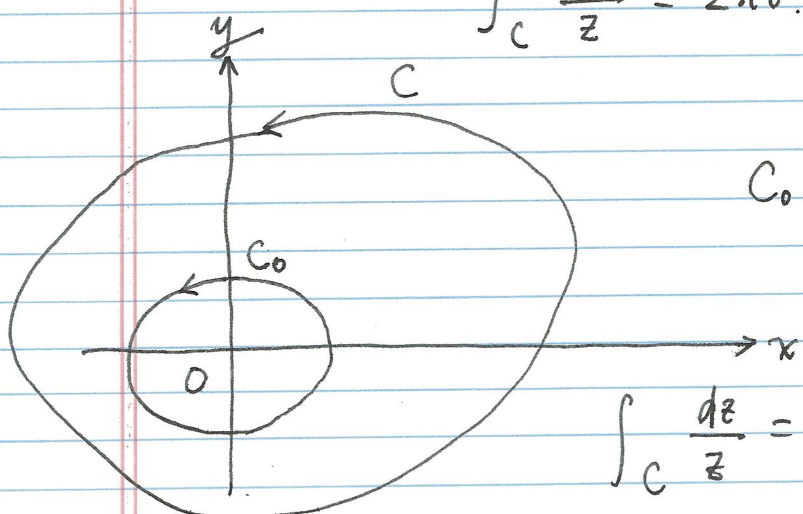
$$\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

$$\Rightarrow \quad \int_{C_1} f(z)\,dz = -\int_{-C_2} f(z)\,dz$$

$$= \int_{C_2} f(z)\,dz.$$

Eg. Let C be any positively oriented simple closed contour surrounding the origin. Then

$$\int_C \frac{dz}{z} = 2\pi i.$$



$C_0: \quad z = Re^{i\theta}, \quad 0 \le \theta \le 2\pi.$

$$\int_C \frac{dz}{z} = \int_{C_0} \frac{dz}{z}$$

$$= \int_0^{2\pi} \frac{Ri\,e^{i\theta}\,d\theta}{Re^{i\theta}}$$

$$= i\int_0^{2\pi} d\theta$$

$$= 2\pi i.$$

Functions of a Complex Variable

More about Contour Integrals

Let $C: \ z = z(t), \quad a \le t \le b$ be a contour. Let $f(z)$ be piecewise continuous on C, i.e. $f(z(t))$ is piecewise continuous on the interval $a \le t \le b$.

The contour integral (or the line integral) of f along C is defined as

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

Properties: $\displaystyle\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$, where z_0 is a complex constant.

$$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

These two properties imply that $\int_C f(z)\,dz$ is linear.

~~[illegible]~~

Let $-C$ ~~[illegible]~~ denote the same contour as C with opposite orientation.

Let $C: \ z(t), \quad a \le t \le b.$

Then $-C: \ z(\tau), \quad b \le \tau \le a$, where $\tau = a + b - t$.

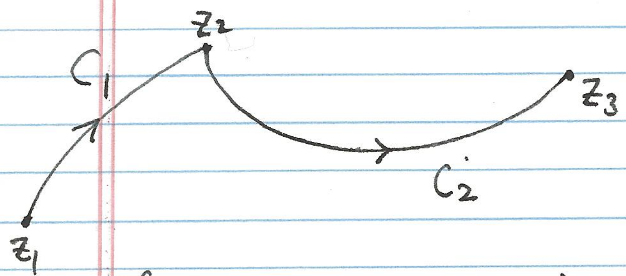
$$\int_{-C} f(z)\,dz = \int_b^a f(z(\tau))\,z'(\tau)\,d\tau$$

$$= -\int_a^b f(z(t))\,z'(t)\,dt$$

$$= -\int_C f(z)\,dz.$$

$$\boxed{\int_{-C} f(z)\,dz = -\int_C f(z)\,dz}$$

Functions of a Complex Variable



Suppose that C is a path that consists of a contour C_1 from z_1 to z_2 followed by a contour C_2 from z_2 to z_3. Suppose that C_1 is represented by

$z = z(t), \quad a \le t \le c$

and C_2 is represented by

$z = z(t), \quad c \le t \le b.$

Then

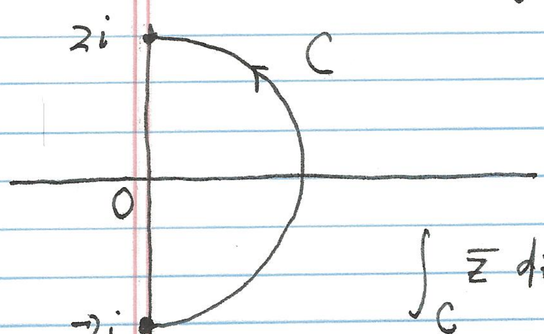
$$\underbrace{\int_a^b f(z(t))\, z'(t)\, dt}_{\int_C f(z)\,dz} = \int_a^c f(z(t))\, z'(t)\, dt + \int_c^b f(z(t))\, z(t)\, dt$$

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz.$$

Eg. Evaluate the integral

$$\int_C \bar{z}\, dz$$

where C is the right-hand half of the circle $|z| = 2$.



$$z = 2e^{i\theta}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\int_C \bar{z}\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{2e^{i\theta}}\, (2e^{i\theta})'\, d\theta$$

$$= 4i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = 4\pi i.$$

Functions of a Complex Variable

$\int_C \bar{z}\, dz = 4\pi i$ can be used to evaluate

$$\int_C \frac{1}{z}\, dz.$$

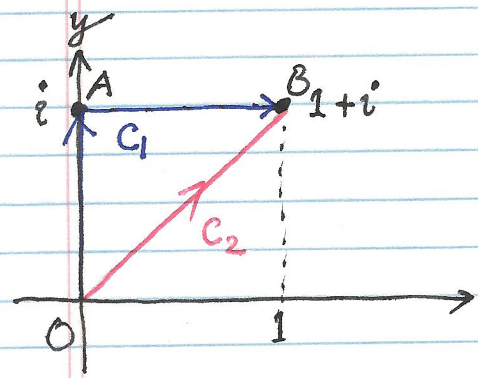
$$\int_C \frac{1}{z}\, dz = \int_C \frac{\bar{z}}{z\bar{z}}\, dz$$

$$= \int_C \frac{\bar{z}}{|z|^2}\, dz$$

$$= \frac{1}{4}\int_C \bar{z}\, dz \qquad (\ |z| = 2\ )$$

$$= \pi i.$$

Eg.



Let C_1 denote the contour OAB and C_2 denote the segment OB. Evaluate the integrals

$$\int_{C_1} f(z)\, dz \quad \text{and} \quad \int_{C_2} f(z)\, dz,$$

where

$$f(z) = y - x - i3x^2 \quad (z = x + iy)$$

Solution. The leg OA can be parametrically represented as $z = 0 + iy$, $0 \le y \le 1$, while the leg AB can be represented parametrically as $z = x + i$, $0 \le x \le 1$. Thus,

$$\int_{C_1} f(z)\, dz = \int_{OA} f(z)\, dz + \int_{AB} f(z)\, dz$$

$$= i\int_0^1 y\, dy + \int_0^1 (1 - x - i3x^2)\, dx = \frac{1-i}{2}.$$

C_2 can be represented parametrically as
$z = x + ix \quad (0 \le x \le 1)$

$$\int_{C_2} f(z)\,dz = \int_0^1 -i3x^2(1+i)\,dx$$
$$= 3(1-i)\int_0^1 x^2\,dx$$
$$= 1-i.$$

Hence, the integrals of $f(z)$ along the two paths C_1 and C_2 have different values. This example shows that the line integral in general depends on the path.

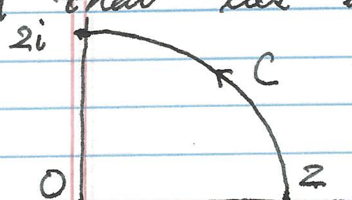
Upper Bounds for Moduli of Contour Integrals

Let C denote a contour $z = z(t)$, $a \le t \le b$.

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$$
$$\le \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

Suppose that $\exists\ M > 0$ such that $|f(z)| \le M$ on C.
Then
$$\left|\int_C f(z)\,dz\right| \le M \underbrace{\int_a^b |z(t)|\,dt}_{\text{the length of the contour}} = ML.$$

Eg. Let C be the arc of the circle $|z| = 2$ from $z=2$ to $z=2i$ that lies in the first quadrant.



Show that
$$\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6\pi}{7}.$$

On C,

$$|z+4| \le |z|+4 = 6 \qquad (|z|=2)$$

$$|z^3 - 1| \ge ||z|^3 - 1| = 7$$

So, $$|f(z)| = \left|\frac{z+4}{z^3-1}\right| \le \frac{6}{7} \text{ on } C.$$

The length L of the contour C is π. Hence,

$$\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6}{7}\pi.$$

Eg. Let C_R be the semicircular path

$$z = Re^{i\theta}, \quad 0 \le \theta \le \pi,$$

and $z^{1/2}$ denote the branch

$$z^{1/2} = \sqrt{r}\,e^{i\theta/2}, \ \left(r>0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the square root function. Show that

$$\lim_{R\to\infty} \int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz = 0.$$

Solution. Suppose that $|z| = R > 1$. On C_R,

$$|z^{1/2}| = |\sqrt{R}\,e^{i\theta/2}| = \sqrt{R}$$

and

$$|z^2+1| \ge ||z|^2 - 1| = R^2 - 1.$$

Consequently on C_R,

$$\left|\frac{z^{1/2}}{z^2+1}\right| \le \frac{\sqrt{R}}{R^2-1}.$$

$$\left| \int_{C_R} \frac{z^{1/2}}{z^2+1}\, dz \right| \le \frac{\sqrt{R}}{R^2-1} \underbrace{\pi R}_{L} \longrightarrow 0 \text{ as } R \to \infty.$$